2.2 FILTERING

2.2a THE BASIC IDEAS OF FILTERING:

In section 2.1, we looked at how certain time waveforms offer rich spectra. We have emphasized however that it is not the details of the spectrum that are important for musical sound production, but rather, the manner in which these details vary in time. Thus we still have before us the task of devising systems that allow us to vary the spectrum. We could consider circuits that cause the waveform itself to vary, and such circuits are used. However, a filter is somewhat more direct, because it alters the spectrum in a predetermined way, and it is the spectrum we are concerned with. With the circuit that causes the waveform to vary in its details, we arrive at a time varying spectrum in general, but essentially this spectrum is unknown, except upon additional Fourier analysis. In the case of the filter, we will always arrive at a time varying spectrum, and the new spectrum is known to us as the component-by-component product of spectral lines times the filters frequency response at the component frequency. For example, Fig. 2-46 shows the product of a sawtooth spectrum times a corner peaked low-pass filter. Once we understand that the filter's frequency response can be made to vary in time, we can understand how the spectrum varies in time as well.

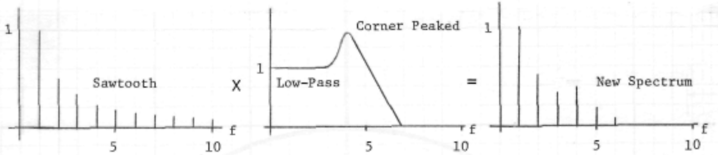


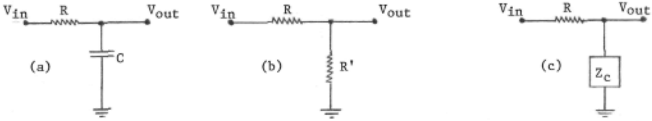
Fig. 2-46 New Spectrum as Product of Sawtooth Spectrum and the Frequency Response of a Corner-Peaked Low-Pass Filter

We will be shortly looking at some of the theory of filtering, and some practical ideas with regard to voltage-controlled filtering. We will be taking a somewhat unconventional approach, omitting much about active filtering, because we have limited space, and the material is available elsewhere. Thus we will be taking the most direct route to filter analysis we can find, and then concentrate on the network structures found in VCF's. The interested reader can find more conventional ideas on filtering in Laboratory Problems and Examples in Active, Voltage-Controlled, and Delay Line Networks, Electronotes Supplement S-016 (1978)

2.2b LAPLACE TRANSFORM NOTATION FOR CAPACITORS:

In order to make a filter in frequency, we need to use a circuit component that is sensitive to frequency, and basically we have the choice of inductors or capacitors. Inductors tend to be bulky, heavy, and non-ideal at audio frequencies, so we will be looking at the capacitor. Our procedure will be to develop the necessary mathematical tools in a manner that will aid intuition, and which will provide a working knowledge of the methods of deriving filter frequency responses. While the level of mathematical rigor will be low, the results are nonetheless correct.

Our first job will be to develop a means of working with capacitors in ordinary network calculations. We will be starting with the circuit of Fig. 2-47a, which is a simple first-order low-pass filter. Fig. 2-47 as a whole defines a transfer of the difficult part of the problem of determining V_{out}/V_{in}. Given Fig. 2-47a, we indicate that we have no way of knowing the function V_{out}/V_{in} as yet. In Fig. 2-47b, we take on a simpler problem since the capacitor C is replaced



$V_{out}/V_{in} = ?$ $\quad$ $V_{out}/V_{in} = R'/(R+R')$ $\quad$ $V_{out}/V_{in} = Z_c/(R+Z_c)$

$Z_c = ?$

Fig. 2-47

by a second resistor R'. In this case, we know exactly how to find the relationship between the input and the output, since we have a simple voltage divider. Now we take the step of supposing that it might be possible to find some value for an impedance which depends on C, which we will call Z_c, and so that we may treat Z_c exactly like a resistor in network calculations. In this case, the relationship between the input and the output is as in Fig. 2-47c. Note that we have not solved the problem, just transferred it. We don't know what Z_c is.

As we warned, the derivation here will be somewhat arbitrary and incomplete. We will find Z_c for a single example, that of applying the step function to the R-C filter. We should also point out that we will be using the Laplace Transform (LT) methods here, and the equating of certain ratios, some of which are in terms of time t, and some in terms of the complex frequency s, is justified on the grounds that we measure frequency responses using sinusoidal waveforms, the common ground between the time and frequency worlds.

Having made the above disclaimer, we will go ahead with the step function problem, as illustrated in Fig. 2-48. What is the output voltage given the step

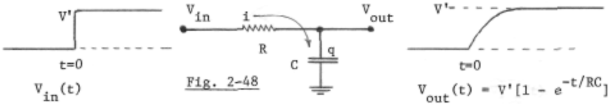


Fig. 2-48

voltage of magnitude V' at the input? You may well know the answer from any number of sources, but we will briefly review the process of obtaining the answer based on a differential equation approach. We need a few physical facts about the system. First, Ohm's Law tells us that the current $i = (V_{in}-V_{out})/R$ flows through R. Secondly, the charge on a capacitor is related to the voltage and capacitance by $q = CV$, so here we have $q = C\,V_{out}$. Thirdly, the current i is the time rate of change of charge, so $i = dq/dt = C\,dV_{out}/dt$. Equating the two expressions for the current i, we get:

$$C\,dV_{out}/dt = V_{in}/R - V_{out}/R \tag{2-44}$$

which is the differential equation of the system. Solution of equation (2-44) is a common exercise, and we will not give the details here. Instead, we give the solution as:

$$V_{out} = V'[1 - e^{-t/RC}] \tag{2-45}$$

which is clearly the correct solution, as can be simply verified by plugging equation (2-45) into equation (2-44), arriving at an identity. The solution is the rising exponential function shown in Fig. 2-48. [As an aside, this is a basic waveshape of AR and ADSR envelope generators, to be studies later.] We now know the answer for one case.

As our next step, we will take the LT of the input and the output, using equation (2-43). In taking the LT, we will be obtaining V_{in} and V_{out} as functions of s, rather than functions of t. The integrations involved in the LT's are simple:

$$V_{in}(s) = \int_0^\infty V_{in}(t)e^{-st}dt = \int_0^\infty V'e^{-st}dt = V'/s \qquad (2\text{-}46)$$

$$V_{out}(s) = \int_0^\infty V_{out}(t)e^{-st}dt = \int_0^\infty V'(1-e^{-t/RC})e^{-st}dt = \frac{V'}{s(sRC+1)} \qquad (2\text{-}47)$$

We can then take the ratio, $V_{out}(s)/V_{in}(s)$ using equations (2-46) and (2-47):

$$V_{out}(s)/V_{in}(s) = 1/(1 + sRC) \qquad (2\text{-}48)$$

Equation (2-48) is the so-called "transfer function" of the network, arrived at by a special case. Usually, we will be using a much simpler method of achieving the transfer function, but for now, we want to get Z_c out of the deal, and to do this, we will refer back to Fig. 2-47c, and equate the voltage divider relationship there to the transfer function (2-48):

$$\frac{Z_c}{R + Z_c} = \frac{1}{1 + sRC} \qquad (2\text{-}49)$$

which is solved for Z_c, giving:

$$Z_c = \frac{1}{sC} \qquad (2\text{-}50)$$

So what does this prove? Well, we have mathematically proven nothing, but we see that if we start with the network, substitute $Z_c = 1/sC$ for C, and solve out the network, treating 1/sC as a resistor, we get the transfer function in the complex frequency domain. Thus treating the impedance of C as a function of the complex variable s, we get rid of differential equations, and get our transfer function using only common network calculations, and algebra. Thus the LT method essentially does our calculus for us, reducing it to algebra.

Nearly everything we want to learn about a given filter network is obtained form its transfer function $T(s) = V_{out}(s)/V_{in}(s)$. After a bit of experience, we obtain T(s) rather rapidly, and then can go on from there to the interesting things. Above we have obtained in our example the transfer function of the first-order low-pass filter, which we can call $T_1(s) = 1/(1+sRC)$. We will have occasion to use this in future developments.

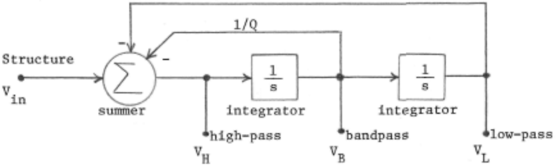
2.2c THE STATE-VARIABLE FILTER:

Two types of voltage-controlled filter (VCF) are common in electronic music synthesizers. One type is the four-pole low-pass, consisting of four first-order low-pass sections in series, with overall feedback. The second type is the state-variable, which has the advantage of offering three filtering functions (low-pass, high-pass, and bandpass) and lending itself very well to voltage-control. The disadvantage of the state-variable is that its low-pass function is only second-order as compared to the four-pole low-pass (fourth-order). Considering the importance of the low-pass function in imitation of the sounds of traditional acoustic type instruments, some synthesizer designers prefer to offer the superior low-pass (four pole) instead of the multiple function filter (state-variable). Overall however, the state-variable is hard to beat for a general application.

The state-variable filter consists of two integrators and a summer, in a structure with feedback, in the manner shown in the block diagram (Fig. 2-49). The LT of an integrator is 1/s, so we will represent the integrators in this way.

Fig. 2-49

State-Variable Structure



Analysis of this basic block diagrams and of all other state-variable filter structures is greatly eased by using the integrator relationship. First the summer output, V_H is clearly the sum:

$$V_H = V_{in} - (1/Q)V_B - V_L \tag{2-51}$$

at the same time, $V_B = (1/s)V_H$ and $V_L = (1/s^2)V_H$, since we just have series integrators acting on V_H. Plugging these into equation (2-51), and solving for V_H/V_{in}, we get:

$$T_H(s) = V_H/V_{in} = \frac{s^2}{s^2 + (1/Q)s + 1} \tag{2-52}$$

which is a typical high-pass filter transfer function. Using the integrator relations, we can get the transfer functions of the two remaining outputs:

$$T_B(s) = T_H(s)/s = \frac{s}{s^2 + (1/Q)s + 1} \tag{2-53}$$

$$T_L(s) = T_H(s)/s^2 = \frac{1}{s^2 + (1/Q)s + 1} \tag{2-54}$$

which are transfer functions of bandpass and low-pass filters respectively. The order of the filter, corresponding to the number of "poles", is the highest power of s in the denominator of T(s), so we have second-order responses (two poles) in all three cases here. [Note that a notch response is also usually obtained by summing the low-pass and high-pass functions].

The transfer function T(s) is really a basic form leading to the frequency response function of the filter. The transfer function also tells us where the poles and zeros of the filter are, and from these positions, we can find the frequency response function in a semi-geometric manner, often useful directly, or to establish an intuitive feeling for the filters operation.

The direct method of finding the frequency response is the following:

$$\text{F.R.} = |T(s{=}j\omega)| = [T(j\omega)\cdot T(-j\omega)]^{\frac{1}{2}} \tag{2-55}$$

What this says is that to get the frequency response, you first substitute into T(s) the value $j\omega$ for s, and then take the magnitude of this expression. Actually, since s is complex, and equal to $\sigma + j\omega$, this "substitution" is really a special case of $\sigma = 0$. Note that T(s) and $T(j\omega)$ are complex functions, but $|T(j\omega)|$, being the magnitude of a complex number, is a real number. Thus equation (2-55) gives us the frequency response function we are used to. Given a frequency f, we can calculate $\omega = 2\pi f$, and then pulg ω into equation (2-55). (Actually, you plug it into the final simplified version). This tells us, for that frequency, the amount by which the filter amplifies or attenuates that particular frequency. A typical case might be obtained from equation (2-54), the low-pass, where the procedure gives for the frequency response:

$$\text{F.R.} = |T_L(s)| = \left[\frac{1}{\omega^4 + [1/Q^2 - 2]\omega^2 + 1}\right]^{\frac{1}{2}} \tag{2-56}$$

We thus see a straightforward method taking shape. First, we replace capacitors with

1/sC (we did not do this for the state-variable filter yet, just assumed 1/s for the integrator - we will look at this more later). Secondly, we solve out the network as though 1/sC were a resistor, arriving at $T(s) = V_{out}(s)/V_{in}(s)$. Thirdly we arrive at $|T(s)|$, the frequency response by using equation (2-55). Thus we start with something physical we know (a circuit with real capacitors, resistors, op-amps, etc.) and through the use of LT notation, we arrive at something we can use and understand, the frequency response.

Note that we do not really use the LT as such in our analysis, but only indirectly [and tabulated LT's are often used in conjunction with T(s), since for an arbitrary but tabulated $V_{in}(s)$, we can obtain $V_{out}(s) = T(s)\cdot V_{in}(s)$ - the results are not limited to V_{in} as sine waves, although the frequency response is usually understood only in terms of sine wave components]. Nonetheless, we have used the complex frequency variable s freely, and should know more about it. First it is a complex number, $s = \sigma + j\omega$, an although we often work only with the $j\omega$ part of s, the σ part has an influence on the results. Thus we look at significant features of a function like T(s) as they occur for all possible values of s, not just for the $j\omega$ part, and we are thus led to look at the so-called "s-plane" or complex frequency plane. This is actually just an ordinary complex number plane, where the real part is the horizontal axis, and the imaginary part is the vertical axis. The significant features of the function T(s) are the "poles" and "zeros." A pole occurs when the denominator of T(s) becomes zero, and hence T(s) blows up to a very large value (a "poles" sticking up in the air!). A zero occurs in a similar manner, when the numerator becomes zero, thus making T(s) zero. Solving for poles and zeros is often trivial, often relatively simple, and at other times, it can be quite a chore. Naturally the difficulty is involved with the order of T(s). For equations (2-52) through (2-54), we find zeros for s=0, s=0, and for no s respectively. Since the zero in equation (2-52) occurs for $s^2=0$, it is a second-order zero (two zeros on top of each other at s=0). Finding the zeros is thus trivial here. Finding the poles is a matter of setting $s^2 + (1/Q)s + 1 = 0$ and solving for s, if by no other mathod, by using the quadratic formula, arriving at:

$$p_1, p_2 = -(1/2Q) \pm j\sqrt{1 - 1/4Q^2} \qquad (2\text{-}57)$$

where p_1 and p_2 are the values of s where the poles occur (with 1 and 2 referring to the + and - signs in front of the square root sign). Study of equation (2-57) will show that if Q is greater than 1/2, the poles are really complex, while if Q is less than 1/2, the poles are real. Most cases of interest will involve complex poles, and this is the reason for writing equation (2-57) with the j showing. Fig. 2-50 shows several things. First, it is an example of the s-plane. Secondly, it shows the position of poles in the s-plane corresponding to several values of Q using equation (2-57). Finally, it shows the position of zeros corresponding to high-pass, bandpass, and low-pass filters.

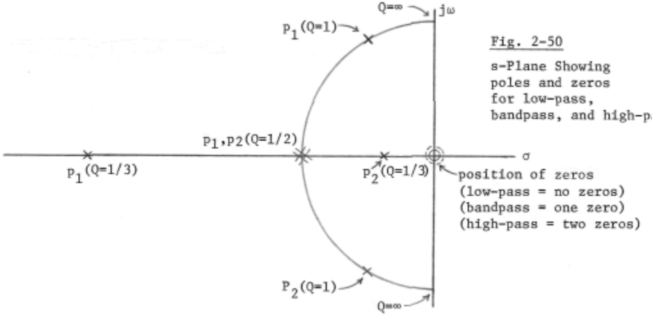


Fig. 2-50

s-Plane Showing poles and zeros for low-pass, bandpass, and high-pass

A number of points should be made concerning Fig. 2-50, which represents the situation not only for the state-variable filter under direct discussion, but also any of the numerous other second-order active filter configurations that are common. [As active filters of fixed frequency and single function, configurations somewhat simpler than the state-variable are common and are recommended. For our purposes, the state-variable is recommended because of its ease in accepting voltage-control, and its providing three functions at once.]. Note first the positions of the poles as a function of Q. As Q exceeds 1/2, the poles become complex conjugate pairs [see equation (2-57)], and they move on a circle from $s = -1$ to $s = \pm j$. That the magnitude of the pole position places it on a circle is verified by calculating the magnitude from equation (2-57):

$$|p_1| = |p_2| = [\ \mathrm{Re}\{p_1\}^2 + \mathrm{Im}\{p_1\}^2]^{\frac{1}{2}} = [(\tfrac{1}{2Q})^2 + (1 - \tfrac{1}{4Q^2})]^{\frac{1}{2}} = 1 \qquad (2\text{-}58)$$

The poles reach $\pm j$ when Q becomes infinite (thus at $\pm j$ where $\sigma = 0$). The value $\sigma = 0$ represents the case of sustained oscillation, and if σ is positive, the output of the system not only oscillates but grows without (theoretical) limit. Thus poles for $\sigma > 0$ represent unstable networks, and thus we have the rule for stability that poles should be in the left half plane. Zeros on the other hand could be anywhere and not affect stability, since a zero is simply a point in complex frequency where the response goes to zero. In our examples, equations (2-52), (2-53), and (2-54), we have zeros that either are not present, or which are at $s = 0$. In fact, we have no problem placing the zeros anywhere we want using the state-variable approach. We just need to add weighted versions of the high-pass, bandpass, and low-pass outputs. Since all three transfer functions have the same denominator (the same poles), we would just get a summed transfer function:

$$T_s(s) = \frac{As^2 + Bs + C}{s^2 + (1/Q)s + 1} \qquad (A,B,C = \text{summing weights}) \qquad (2\text{-}59)$$

Now the numerator of the transfer function is a general second-order function, as is the denominator, and the positions of the zeros are available using the quadratic formula, as for equation (2-57), we get:

$$z_1, z_2 = -(B/2A) \pm (1/2A)\sqrt{B^2 - 4AC} \qquad (2\text{-}60)$$

One case of special interest is that where $A = C = 1$, and $B = 0$, in which case we get zeros as $\pm j$, and a resulting notch filter.

A general transfer function, as in equation (2-59), can be written either in its quadratic form, or in factored form:

$$T_s(s) = \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)} \qquad (2\text{-}61)$$

where equations (2-59), (2-57), and (2-60) have all become involved. If we now ask about the frequency response of the filter in terms of the magnitude of $T_s(s)$, we can write:

$$|T_s(s)| = \frac{|(s - z_1)|\cdot|(s - z_2)|}{|(s - p_1)|\cdot|(s - p_2)|} \qquad (2\text{-}62)$$

Keeping in mind that all the quantities in equation (2-62) are complex numbers, we have magnitudes of the differences between complex numbers, and these are just the distances in the complex plane. Thus equation (2-62) is actually saying that to get the frequency response $|T_s(s)|$, we can first determine the positions of the poles and zeros, and than get $|T_s(s)|$ as the product of the distances from s to the zeros, divided by the product of the distances from s to the poles. This is true for any s, and in particular for $s = j\omega$, and thus when we choose s as a point $j\omega$ on the imaginary axis, we are doing the equivalent of equation (2-55). The result is the same frequency response. The principle is illustrated in Fig. 2-51. This geometric interpretation is useful where the poles are known, and can even be used to solve for the frequency response graphically (actually measuring distances on a graph of the complex plane). In other cases, the geometric interpretation can be used to lead to

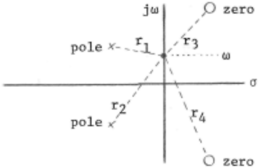


Fig. 2-51

Graphical Interpretation of $|T(s)|$

$$|T(s)| = \frac{r_3 \cdot r_4}{r_1 \cdot r_2}$$

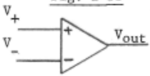
a closed form expression for the frequency response based on known geometry, instead of just working point-by-point along the jω-axis as is illustrated for a single point in Fig. 2-51.

A further consideration of the geometric interpretation allows us to get a rough idea of the frequency response by just looking at the positions of the poles and zeros. Clearly if we want to know the response, we must travel along the jω axis from zero on up (we could go down, but everyone goes up!). If on this trip we happen to pass near a pole, the response will peak in that region. If we happen to pass near a zero, the response will drop. If we find a zero on the jω-axis, actually run into it instead of just getting close, there will be a complete cancellation or notch in the response. If we find a pole on the jω - axis instead of just getting close, we get oscillation, because the response in infinite, and even for a very small input (which is always present as noise in a practical setup), the response will start up. Some interpretations, leading to various filter response functions, are shown in Fig. 2-52.

In order to understand actual realizations of the state-variable filter, or just about any active filter for that matter, it is necessary to understand the operational amplifier or op-amp. The op-amp is basically a high gain differential amplifier. We generally assume the op-amps in active filter networks to have various ideal properties, many of which are closely approximated by real op-amp integrated circuits. One such ideal assumption is that the inputs to the op-amp draw no current, but instead respond to a voltage applied to them without changing that voltage in any way. This is really saying that they have infinite input impedance. A similar assumption with regard to the output is that the output impedance is zero. Both of these approximations are quite good in the case of active filter networks built with modern field-effect transistor inputs of various types. These assumptions are easily placed in the back of one's mind, and taken for granted.

A more important assumption from a design and analysis point of view is the one that considers the amplifier to have infinite gain. Most other ideal assumptions tell us what the op-amp does not do, but this ideal assumption tells us how the op-amp actually behaves. Fig. 2-53 shows the usual symbol for an op-amp along with its equation. The infinite gain assumption is that A goes to infinity. This may seem like a complicating assumption, but actually, it simplifies things, as there are only two possibilities: either the output is pinned against one of the power supply levels, or the input must be zero. The cases of interest are those where negative feedback is present, thus automatically zeroing the input voltage. We will look at two op-amp setups that are found in the state-variable filter.

Fig. 2-53



$$V_{out} = A(V_+ - V_-)$$

$$A \to \infty$$

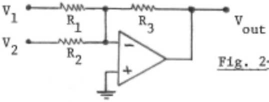
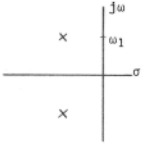


Fig. 2-54

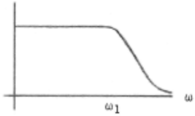
$$V_{out} = -V_1(R_3/R_1) - V_2(R_3/R_2)$$

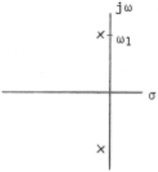
Fig. 2-52 Filter Types, Pole/Zero Plots, and Interpretations

Pole/Zero Plot | Description | Frequency Response

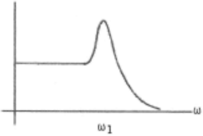


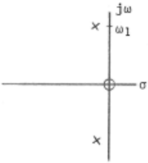
LOW-PASS formed with two complex poles, no zeros, and poles not too close to jω-axis. Response begins to roll down as ω passes by the upper pole, and as ω gets far from the two poles (high frequency), the response continues "downhill". You can consider this is due to two zeros at infinity if you like.



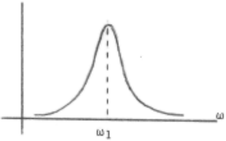


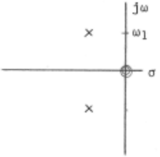
CORNER-PEAKED LOW-PASS is formed by raising the Q, bringing the poles closer to the jω-axis. This causes the Low-Pass to take on more of a Bandpass response as Q increases. Peaking begins when Q exceeds its maximally flat "Butterworth" value of $1/\sqrt{2}$.



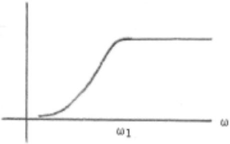


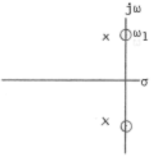
BANDPASS is formed by adding to the poles a zero at s = 0. This means that the response at DC is zero. Moving away from the zero at zero up to the "high ground" near the pole, the response peaks. Beyond this pole, the response drops again (one zero still at infinity if you like).





HIGH PASS is formed with two zeros at s = 0 instead of just one. Again we have zero response at DC, and an increasing response as the pole is approached more closely, but the response does not drop back at high frequency. Looking back from high frequency, the poles and zeros tend to cancel.





NOTCH is formed by realizing complex zeros right on the jω-axis. When the frequency hits the zero, the response goes to zero. The poles serve to support the response on either side of the notch, making the notch sharper as pole Q increases.

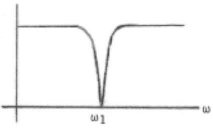


Fig. 2-54 shows the implementation of a summer (inverting summer in this case). The circuit has a negative feedback link, R_3 running from the output back to the (-) or inverting input. Thus we will assume that the input voltage to the op-amp itself, meaning here the differential input between (+) and (-) inputs, is zero. [More on this later.] Thus the (-) input and the (+) input must be at the same voltage. The (+) input is grounded, and thus the (-) input must also be at ground potential. This arrangement, (+) input grounded with negative feedback operating, is commonly encountered, and the (-) input in such a case is called a "virtual ground" or a "summing node." With the (-) input assumed at ground, we can find three currents through R_1, R_2, and R_3 as V_1/R_1, V_2/R_2, and V_{out}/R_3 respectively. No current flows into the actual (-) input of the op-amp, and thus the currents through the three resistors must sum to zero. Setting the sum to zero and solving for V_{out}, we get the sum as shown in the equation of Fig. 2-54. A few words should be said about how the (-) input is maintained at zero voltage. Consider that if any one of the three voltages V_1, V_2, or V_{out} in Fig. 2-54 should go up for any reason, the voltage at the common junction point, the (-) input, must go up as well by some amount. This would cause the (-) input to become positive relative to the (+) input, and the very high (assumed infinite) gain of the op-amp would cause the output V_{out} to move in the negative direction. On the one hand, this tells us why the output takes on the value indicated by the equation in Fig. 2-54, and on the other hand, it tells us why the output responds to the changes in V_1 and V_2.

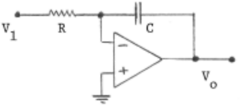


Fig. 2-55
Integrator

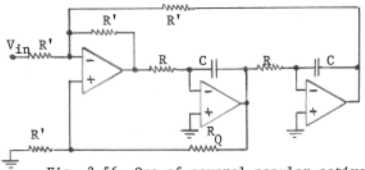


Fig. 2-56 One of several popular active filter configurations of state-variable

Fig. 2-55 shows the second op-amp setup that is part of a state-variable filter: the integrator. We will analyze this to illustrate several principles. First we will show how our use of Z_c [from equation (2-50)] can be used to get the response of the integrator with little trouble. Recall that we have said that we can replace C in any network with 1/sC, and treat it just as we would a resistor. This done, we can see that Fig. 2-55 is a special case of Fig. 2-54, with $R_1 = R$, $R_2 = \infty$, $R_3 = 1/sC$, and $V_{out} = V_o$. We then apply the summer equation from Fig. 2-54 to get:

$$V_o = -V_1\left(\frac{1/sC}{R}\right) = -V_1(1/sCR) \qquad (2\text{-}63)$$

If we now wish to describe a transfer function for the circuit, we can call this $T_1(s)$ and write:

$$T_1(s) = V_o(s)/V_1(s) = -1/sCR \qquad (2\text{-}64)$$

This is the transfer function of a negative integrator. As we said above, the integrator is represented in LT notation by 1/s, and here we have 1/sCR. Thus our earlier example has to be considered a "normalized" case where RC = 1. We will discuss this a bit more later.

We now have all the elements of an active state-variable filter. If we want to realize the block diagram of Fig. 2-51, we can do it. One problem would be that the integrator we have looked at is inverting. The cascade of two inverting integrators is thus non-inverting, but the output V_B is inverted. We could use an inverter (Fig. 2-54 with $R_1 = R_3$, $R_2 = \infty$), but this would add an op-amp. A three op-amp version is still possible, as in Fig. 2-56, where the negative V_B

is fed back to the (+) input of the summing op-amp, thus achieving the needed negative input to the summer. With this voltage divider on the (+) input of the summing op-amp, the (-) input is not at virtual ground. Yet it is still true that the (-) input and the (+) input are at the same voltage, and the analysis goes forward on that basis. We will not pursue this further here however.

We need to say a few words about the denormalization of the transfer function so that we can not only set the shape of the frequency response, but also set the cutoff frequency (or other characteristic frequency point) where we want it. To do this, we can go back to Fig. 2-49, and instead of 1/s for the integrator, use 1/sCR, a positive integrator, which is what we can often obtain using voltage-control elements. This done, we would replace equation (2-52) with:

$$T_H(s) = \frac{s^2}{s^2 + (1/Q)(1/RC)s + 1/R^2C^2} \tag{2-65}$$

and equations corresponding to (2-53) and (2-54) are achieved by dividing equation (2-65) by 1/sRC and $1/s^2R^2C^2$ respectively, achieving:

$$T_B(s) = \frac{s/RC}{s^2 + (1/Q)(1/RC)s + 1/R^2C^2} \tag{2-66}$$

$$T_L(s) = \frac{1/R^2C^2}{s^2 + (1/Q)(1/RC)s + 1/R^2C^2} \tag{2-67}$$

These equations are denormalized, and also show us a "dimensionality" that was not obvious in the earlier normalized equations. That is, the dimensions of s, a frequency, are 1/time. Also, the product of a resistor times a capacitor, RC, is a time. Thus we can see that numerator and denominator in all the transfer functions of equations (2-65), (2-66), and (2-67) have dimensions of $1/\text{time}^2$.

In attempting to arrive at a characteristic frequency, we need to realize that s is really the frequency variable. All sorts of exact and elaborate relationships can be worked out, but we will not do this here. Instead, we will just observe that the characteristic frequency is $\omega = 1/RC$, or $f = 1/2\pi RC$. By characteristic frequency we mean generally, the peak of the bandpass response, and a frequency in the vicinity of the cutoff of the low-pass and high-pass responses. If we solve for the poles of equations (2-65), (2-66), and (2-67), we find that they are now on a circle of radius 1/RC when they are complex, instead of a radius of 1 as was the case with the normalized equations [see equation (2-58)]. Thus if the poles are complex and of a Q somewhat in excess of 1/2, they lie somewhere in the vicinity of the point $\omega = 1/RC$ on the imaginary axis. This is why significant changes in the response function, which occur near the poles (or zeros), occur near 1/RC, and why we find characteristic frequencies near 1/RC, or closely related to it. Since we find it fairly easy to change resistors, either manually with a pot, or through the use of voltage-controlled resistors, it is by varying R that we control the frequency of the filter.

In controlling the R of the integrator, we can use a pot (Fig. 2-57a) for manual control, but for voltage control, we can consider the use of a multiplier (Fig. 2-57b) or a transconductance multiplier (Fig. 2-57c). The use of the multiplier and the transconductor are quite similar. However, the multiplier we have in mind is a four-quadrant voltage-output type, and thus requires a voltage to control it, and a resistor as shown (Fig. 2-57b). A simple, less expensive, and more direct approach is to use the transconductor (Fig. 2-57c). The transconductor is a two-quadrant multiplier, has a current output (avoiding the integrator resistor), and is current controlled (as we shall see, voltage-control is what we have externally, but because of the need for exponential control functions, modules are internally current rather than voltage-controlled). Thus we might expect that the transconductor is acting in

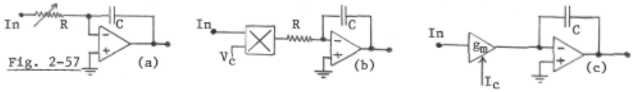


Fig. 2-57

the manner of a voltage-controlled resistor. In fact, we can look at the equation of a transconductor, or OTA (Operational Transconductance Amplifier) as they are also called. A typical equation would be:

$$I_{out} = K\cdot(V_+ - V_-)\cdot I_c \qquad (K = \text{constant}) \qquad (2\text{-}68)$$

Let's suppose that the OTA is being used to form a voltage-controlled integrator as in Fig. 2-58. Here the (+) input has been grounded, so equation (2-68) becomes:

$$I_{out} = -K\cdot V_{in}\cdot I_c \qquad (2\text{-}69)$$

This same current I_{out} must be flowing through the capacitor C, and treating this as a resistor 1/sC, and at the same time realizing that the (-) input of the op-amp is at virtual ground, the output of the op-amp must take on the proper value to "collect" the current I_{out}. Thus a simple extension of Ohm's law gives us:

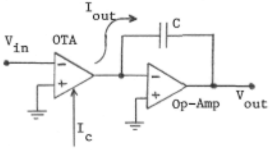


Fig. 2-58

$$V_{out} = -(I_{out})(1/sC) = -(-K\cdot V_{in}\cdot I_c)(1/sC) = V_{in}[(KI_c)/sC] \qquad (2\text{-}69)$$

and this may be written as:

$$T(s) = V_{out}/V_{in} = 1/sCR_e \qquad (2\text{-}70)$$

where: $R_e = 1/KI_c$ (2-71)

is an equivalent resistance for the OTA. Thus the characteristic frequency of a state-variable filter formed from OTA integrators would be:

$$\omega = 1/R_eC = K\cdot I_c/C = (K/C)\cdot I_c$$

Thus we have the frequency of the filter controlled directly by a controlling current I_c. Since the OTA offers both (+) and (-) inputs, we are able to form here a positive integrator. We also have a negative summer (Fig. 2-54) to fill out the basic block diagram (Fig. 2-49), and a final basic state-variable VCF would have the form of Fig. 2-59.

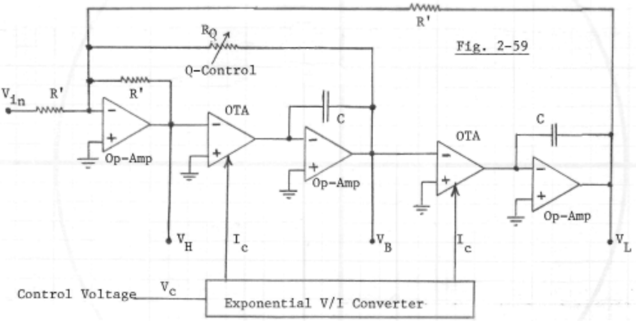


Fig. 2-59

The state-variable VCF should be pretty well understood at this point, with the possible exception of the function of the Q control. Comparing Fig. 2-49 with Fig. 2-59, and using the summer of Fig. 2-54, we are able to identify the quantity Q with the inverse of the feedback gain from the bandpass output back to the input, and this gain is R'/R_Q, so the $Q = R_Q/R'$. Thus we can set the Q as desired by making R_Q a pot as shown. Note that Q is limited by the maximum resistance of the pot. It would also be good practice to add a resistor in series with R_Q to set a minimum value of Q. A value for this series resistor of $R'/2$ might be a good choice, making Q take on values of 1/2 or greater. What is the physical meaning of Q? This is shown in Fig. 2-60 which shows the bandpass response. If we graph the response and determine three frequencies, f_c, f_ℓ, and f_u, (the center frequency, lower 3db frequency, and upper 3db frequency respectively) then Q is equal to the center frequency divided by the 3db bandwidth, thus:

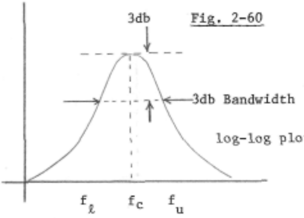


Fig. 2-60

$$Q = \frac{f_c}{f_u - f_\ell} \tag{2-72}$$

Thus Q is a measure of the sharpness of the bandpass function. Note however that as Q goes up past $1/\sqrt{2} = 0.7071...$ in the second-order case, the low-pass and high-pass functions start to peak at the corners. For a high enough Q, all three functions, low-pass, bandpass, and high-pass are sharply peaked and look very much like the bandpass function.

2.2c <u>THE FOUR-POLE LOW-PASS VCF</u>:

We have seen with the state-variable how the numerator of the filter's transfer function pretty much determines the basic filter function, and the the Q determines the details of the behavior in the vicinity of center frequencies or cutoff frequencies. There is one more area to consider, and this is the final roll-off rate. That is, in the area where the signal is being rejected, how fast is this rejection changing with frequency? This is where the number of poles will have a major effect.

How fast does the state-variable low-pass roll off at high frequencies? This we can easily answer using equation (2-56), which for large ω becomes $1/\omega^2$. Thus if ω doubles, a one-octave change, we would get a change in response in decibels of:

$$20 \text{ Log}_{10}\left[\frac{1/\omega}{1/(2\omega)^2}\right] = 20 \text{ Log}_{10} 4 = 12.04 \text{ db} \tag{2-73}$$

and thus we say that the second-order filter rolls off at 12 db per octave. We can attribute this to the major term of the denominator at high frequency, which is the s^2 term in this case. It is not difficult to show in a manner similar to that of equation (2-73) that if the highest term in the denominator is s^n, then the final or "asymptotic" roll-off goes as $6n$ db/octave. Put another way, in an octave distance at high frequency, a single pole filter (s in the denominator) falls by a factor of 1/2, while a two pole filter (s^2 in the denominator) falls by a factor of 1/4, and an n-pole filter (s^n in the denominator) falls by a factor $1/2^n$.

A two-pole filter, falling by a factor of 1/4 per octave is adequate for many purposes, but not all that spectacular. Fixed frequency active filters of 4th, 6th, 8th, and even 10th order and higher are sometimes encountered. Thus we might look for another approach other than the state-variable. In fact, a different approach, the four-pole low-pass, actually came first, being the principal filter used in the early Moog synthesizers. This filter, being four-pole, offers 24 db/ octave, or a factor of 1/16 drop in one octave. Yet just getting four poles total