SECTION 2

SUBTRACTIVE SYNTHESIS

2.0 Introduction:

In this section, we will be looking at subtractive synthesis. There are several reasons for looking at subtractive synthesis at this point. First, there is the fact that the technique is very common, and easy to implement with relatively simple electronics. Secondly, there are several techniques that we will introduce here which will be important later (for example, Fourier analysis and the use of controlled gain blocks). Finally, we have to start somewhere, and have mainly the choice of additive or subtractive synthesis. We feel that additive synthesis is best viewed in its proper perspective when the capabilities of subtractive synthesis are understood.

The power of the subtractive synthesis method can be understood from the fact that on the one hand, selection of a wide bandwidth signal gives us a lot of frequency components, and on the other hand, a single filter can process a large number of components at one time. Thus we choose two efficient processes at the heart of our method. The selection of the waveforms and filters we are going to use is, as with most areas of engineering, a matter of practicality as much as anything. We choose waveforms that are convenient to generate, even though they may not be exactly the best choice otherwise. We determine their harmonic content using Fourier analysis, so we know what we are starting with. It may be as important to know what we are starting with as it would be to be able to choose the starting waveform arbitrarily. Practicalities also dictate our choice of filters. We must choose filters that have the desirable features that we need, and we must choose them so that we can implement voltage-control. Often the design of a filter results in a configuration determined by convenient voltage-controlled filter building blocks.

In order to begin our study of subtractive synthesis, we need to first look at two important topics: Fourier analysis, and filter theory.

2.1 Fourier Analysis Of Waveforms:

2.1a WAVEFORMS AS THE SUPERPOSITION OF SINUSOIDAL COMPONENTS:

The reader perhaps knows already, or has learned from Section 1, that periodic waveforms can be thought of in terms of their frequency spectrum, as a collection of sine waves. A sine wave is the most fundamental of all waveforms. The exact reason as to why this should be so is really a philosophical question, because there are in fact not just other choices possible, but in fact an infinite number of other choices. What we do know is that many physical systems choose as their solution a performance that can be described as a sine wave. [A mass on a spring moving horizontally on a frictionless table. Light waves. Etc.] That they choose sine waves as their basic mode is no surprise if we consider that their performance is determined by differential equations of a certain type. So the real question has more to do with why physical laws are such that certain differential equations appear. Other systems have solutions that can be considered to be (or may <u>be</u>) the superposition of sine waves of different frequencies and phase. Whether or not a waveform is actually composed of sinusoidals, or is just equivalent to a set of superimposed sinusoidals, is another of those philosophical questions. The above should be kept in mind, not so much for the way it may help our analysis, but because the technically trained worker in electronic music will be asked these questions sometime in his career. It is well to know that the answer is not simple.

So we set aside any questions as to why we do things this way, and take the pragmatic view that it is operationally convenient. Others do their math this way and so therefore shall we. Perhaps there is something fundamental still to be discovered. Also, it is probably not incorrect to say that breaking the waveform into sinusoidals is the way the ear does it too.

In Fig. 1-8 of Section 1, we saw how a waveform is formed by adding harmonics, or to take the other point of view, is actually composed of and can be broken down into, the same harmonics. To generate these diagrams, it was necessary to know the Fourier Series (FS) of the sawtooth waveform. Working by trial and error would have been difficult, as even if we had hit upon the correct amplitudes for the harmonics (falling as 1/n where n is the harmonic number), thus achieving essentially the sawtooth's musical timbre, we would still have to have arranged all the phases correctly to get the sawtooth shape.

2.1b A CHOICE OF WAVEFORMS FOR SUBTRACTIVE SYNTHESIS

The sawtooth's musical timbre (tone "color" if you will) is useful musically as it is "full" and "bright" to use subjective terms. Roughly, full means that no harmonics are missing, and bright means that the higher harmonics contain a good portion of the energy of the whole waveform. Since the sawtooth's timbre is useful, we know that a waveform with harmonics falling off as 1/n is useful, but because the ear is essentially phase deaf, any waveform with harmonics falling off as 1/n will do, and sound essentially the same as the sawtooth. So why not use some other such waveform? Because the sawtooth <u>is</u>. It already exists, and is easy to generate. About all we have to do is arrange to create a voltage ramp and then reset this ramp to some initial value when it exceeds an upper limit.

Thus we tend to start with waveforms that are easy to generate, and these are the ones with relatively simple waveform geometry. Typical waveforms are the saw, triangle, square, and pulse, which will be discussed in detail. Notice that the sine wave is not a waveform of simple geometry. In fact, the sine wave is not a simple waveform to generate, especially if we require one of very high purity. The usual method is to shape the triangle wave into a sine approximation. As with the case of filters as mentioned in the introduction to this section, many of our design choices for oscillators are influenced strongly by our ability to implement voltage-control. Accuracy of control is much greater when only one control element needs to be varied. Most true sinewave generators would require two control elements in parallel to be varied. Also, we do not really want to build a separate oscillator for each waveform we need. Instead we prefer to generate accurately one waveform of simple geometry (almost always the saw or triangle) and then waveshape this to the others we need. Shaping of a sinewave into other waveforms is very difficult. Finally, add to this the fact that a sinewave of very high purity is of very limited musical value in general, and of even less value in subtractive synthesis (nothing to filter out), and we see that approaches to Voltage-Controlled Oscillators (VCO's) based on triangle or saw are the only rational design choices.

The table of Fig. 2-1 shows the usual set of waveforms available on a voltage-controlled synthesizer. Two basic types of VCO are common, the sawtooth based design, and the triangle-square based design. In the sawtooth based design, the major task is to convert the sawtooth to triangle. The triangle can be rounded with a non-linear circuit into a sine, and the square and pulse waveforms are obtained from either the saw or the triangle using comparators. The square is of course a special case of the pulse where the "duty-cycle" is 50-50. Most designs will make the pulse width variable (usually voltage-variable) by adjusting a "reference" level on the comparator. In a triangle-square based VCO design, the triangle and square waveforms are available directly from the oscillator. The triangle is rounded to sine, and pulses can be obtained from the triangle with a comparator. This leaves the triangle-to-sawtooth as the only major problem to consider. These are the major waveforms that are expected, and we shall shortly subject them to Fourier analysis.

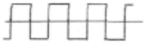
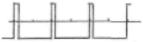
WAVEFORM	SHAPE	HARMONIC CONTENT	SUBJECTIVE TIMBRE
Sawtooth		all harmonics falling off as 1/n	full, bright
Triangle		all odd harmonics, falling off as $1/n^2$	thin, mellow
Square		all odd harmonics, falling off as 1/n	hollow, edgy
Pulse		all harmonics, falling as 1/n, but with additional lobed structure	full, bright, edgy
Sine		*fundamental only*	*very mellow*

Fig. 2-1 TABLE OF COMMON WAVEFORMS

The waveforms in the table of Fig. 2-1 offer a wide variety of spectral content, but we can always find room for a few other possible ones. In particular, there does not seem to be a waveform with only even harmonics. Well, if we think about it we see that only even harmonics would mean that frequency components would be 2f, 4f, 6f, 8f, and so on. This is exactly the same as all harmonics of 2f, so nothing new is gained. Thus the first interesting case of even harmonics is one with a fundamental and then all even harmonics. Thus it would contain f, 2f, 4f, 6f, 8f, and so on. If such a waveform is desired, it is available as the half-wave rectified sine.

To summarize, any oscillator (VCO) that is to be used in subtractive synthesis will probably have available various waveforms of varying harmonic content, and these are chosen based mainly on convenience of generation. Typical waveforms are sawtooth, triangle, square, pulse (variable width), and sine (approximation). Their basic properties are summarized in Fig. 2-1.

2.1c THE FOURIER SERIES:

The Fourier Series (FS) of a periodic waveform gives us its spectrum. It is a formal mathematical development that can take on various forms, all with more or less the same result of allowing us to say, for example, that the spectrum of a square wave consists of all odd harmonics falling off as 1/n. In musical work, this is pretty much all we need. The FS strictly applies only for waveforms that are perfectly periodic and which started at time $-\infty$, and which will continue to time $+\infty$. There are of course no such waveforms, as we are accustomed to having our waveforms start and stop over relatively short periods of time. Thus we are mainly interested in the FS as it may apply in an approximate way. We are interested in the FS as a spectrum determining tool, and we are interested in the spectrum as it is perceived by the ear. The ear has a limited time constant - a time after which the perception fades, typically on the order of 0.05 seconds. Thus if we start a periodic waveform, after about 0.05 to 0.2 seconds, the ear has forgotten that the tone actually began only recently - it may have been going on forever! Likewise, the ear does not know that the tone will stop. During this time there is a good deal of validity to the FS approach to the spectrum calculation.

We should mention that another Fourier technique, the Fourier Transform (FT) is the proper one to apply to signals that are not infinite in both durations. A FT model can be used to tell us a good deal about what we hear while a periodic waveform is being started. We can understand the "click" that we hear if the tone comes on abruptly in terms of the wide bandwidth of the spectrum during this transition time.

Thus the FT model is probably more realistic for the ear than the FS model. However, the FS model being simpler and still very useful is the one we will choose to look at first.

The musical engineer will not often have occasion to actually calculate out a FS from scratch. He should know however how to use tabulated series, and enough about the process to work out the untabulated cases when he has to. Thus we will go over one calculation as an example. We will choose the 1/3 duty cycle pulse. We will not derive the formulas for the FS, which are derived in just about any book on engineering mathematics. The FS equation and the equations for the coefficients a_n and b_n are given below:

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} [a_n \text{Cos}(n\pi t/L) + b_n \text{Sin}(n\pi t/L)] \qquad (2\text{-}1)$$

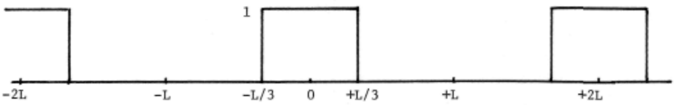
$$a_n = \frac{1}{L} \int_{-L}^{L} f(t)\ \text{Cos}(n\pi t/L)dt \qquad n = 0, 1, 2, \ldots \qquad (2\text{-}2)$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t)\ \text{Sin}(n\pi t/L)dt \qquad n = 1, 2, 3, \ldots \qquad (2\text{-}3)$$

Here f(t) is the periodic function to be expanded in a FS, one period of this function is of length 2L, n is the order of the harmonic, and a_n and b_n are the FS coefficients. There are quite a few other equivalent equations for these above, and there is also a complex form of the FS arrived at through the use of the Euler identity $e^{j\theta} = \text{Cos}\,\theta + j\ \text{Sin}\,\theta$. In the complex form, there is only one coefficient (usually c_n) and thus we have only a pair of equations, and a relatively easy conceptual step to the FT. Here however we will use only the series above.

The 1/3 duty cycle pulse is shown in Fig. 2-2. The spacing of the pulse within the interval -1 to +1, or in any other interval of length 2L, is arbitrary as far as the final line spectrum is concerned. Naturally we choose a spacing that will lead to the simplest math, and experience shows that if we choose an initial spacing that has a degree of symmetry, we have a good chance of getting some of the coefficients (usually either all the a_n or all the b_n) to cancel out. This will be discussed as the example proceeds.

Fig. 2-2 f(t) = 1/3 duty cycle pulse



The equations for the coefficients involve integrals, and in general in order to form the FS, we must be able to integrate f(t)Cos(nπt/L) and f(t)Sin(nπt/L), which is usually fairly simple if f(t) is a simple function. Here f(t) is a pulse taking on only the values 0 and +1. This means that we need only be able to integrate the Sine and Cosine functions themselves, and that the integral is zero over part of its limits, so we can bring the limits in. Thus for our example, equations (2) and (3) become:

$$a_n = \frac{1}{L} \int_{-L/3}^{L/3} 1 \cdot \text{Cos}(n\pi t/L)dt \qquad (2\text{-}4)$$

$$b_n = \frac{1}{L} \int_{-L/3}^{L/3} 1 \cdot \text{Sin}(n\pi t/L)dt \qquad (2\text{-}5)$$

These are easy to integrate if we change them to the form:

$$a_n = \frac{1}{n\pi} \int_{-L/3}^{L/3} \text{Cos}(n\pi t/L)\, \frac{n\pi}{L} dt \tag{2-6}$$

$$b_n = \frac{1}{n\pi} \int_{-L/3}^{L/3} \text{Sin}(n\pi t/L)\, \frac{n\pi}{L} dt \tag{2-7}$$

which become:

$$a_n = \frac{1}{n\pi} \left[\text{Sin}\,\frac{n\pi t}{L}\right]_{-L/3}^{L/3} = \frac{1}{n\pi}\left[\text{Sin}(n\pi/3) - \text{Sin}(-n\pi/3)\right] \tag{2-8}$$

$$b_n = \frac{-1}{n\pi} \left[\text{Cos}\,\frac{n\pi t}{L}\right]_{-L/3}^{L/3} = \frac{1}{n\pi}\left[\text{Cos}(n\pi/3) - \text{Cos}(-n\pi/3)\right] \tag{2-9}$$

Now, because the Sine is an odd function [Sin(-x) = -Sin(x)] and the Cosine is an even function [Cos(-x) = Cos(-x)], we find that equations (2-8) and (2-9) become:

$$a_n = \frac{2}{n\pi} \text{Sin}(n\pi/3) \tag{2-10}$$

$$b_n = 0 \tag{2-11}$$

Thus we have only the a_n coefficients, and we can calculate them using equation (2-10) which gives the a_n in terms of n.

There could be a special problem in connection with a_o. This can be avoided by doing a separate integration, where the Cos(nπt/L) term in equation (2-4) becomes 1 for n=0, and we get from equation (2-4) $a_o = 2/3$, and thus from equation (2-1), the DC term is $a_o/2 = 1/3$, which is the appropriate DC term for a 1/3 duty cycle pules. The problem with using equation (2-10) is that when n goes to zero, both n in the denominator and Sin(nπ/3) in the numerator go to zero, and it is necessary to take the limit, as otherwise we have the meaningless 0/0. Fortunately in this case it is a well known limit, as can be seen by writing equation (2-10) for n = 0 as:

$$a_o = \lim_{n\to 0} \frac{2}{3}\, \frac{\text{Sin}(n\pi/3)}{n\pi/3} \tag{2-12}$$

and it is well known that the limit as x goes to zero of Sin x/ x is one. Thus we get $a_o = 2/3$ as by the direct method. So the equation (2-10) is correct even for n=0 as it should be. Equation (2-10) is also obviously equal to zero if n is an integer multiple of 3, since in this case Sin(nπ/3) = Sin(mπ) = 0. Using equation (2-10), we can calculate other a_n as given below:

n	a_n
0	2/3
1	0.5513
2	0.2757
3	0
4	-0.1378
5	-0.1103
6	0
7	0.0788
8	0.0689
9	0
10	-0.0551
11	-0.0501
12	0

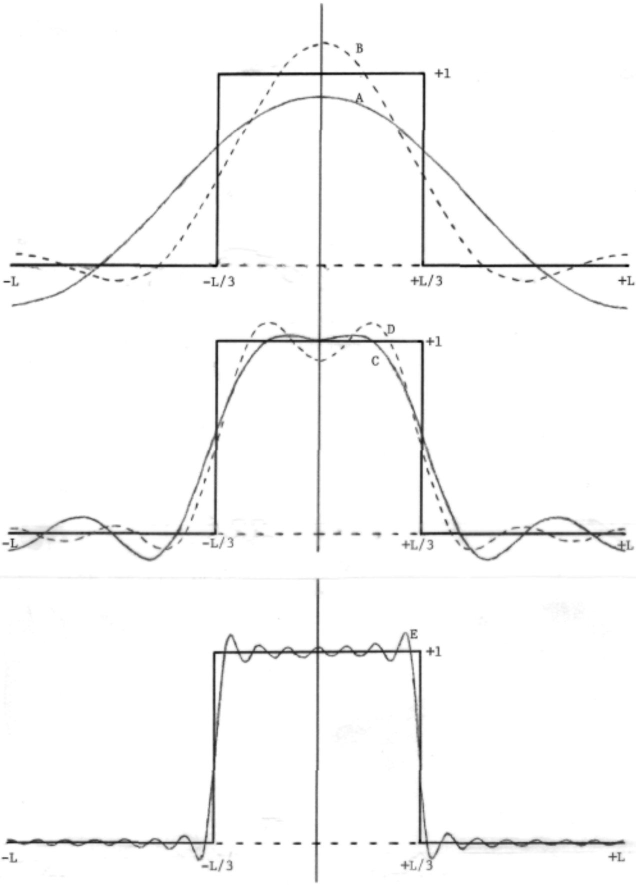


Fig. 2-3 Approximations to 1/3 duty cycle pulse. A: First harmonic only. B: First and second harmonics. C: First, Second, and fourth harmonics. D: First, second, fourth, and fifth harmonics. E: Harmonics 1,2,4,5,7,8,10,11,13,14,16,17,19, and 20. All also include the DC term of 1/3.

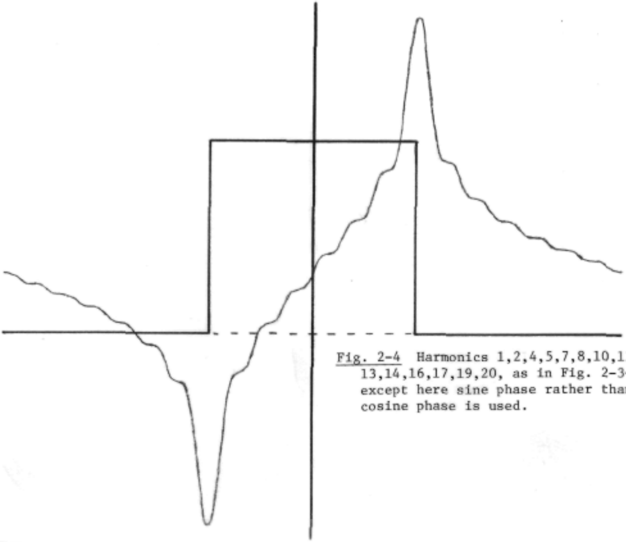


Fig. 2-4 Harmonics 1,2,4,5,7,8,10,11, 13,14,16,17,19,20, as in Fig. 2-3-E except here sine phase rather than cosine phase is used.

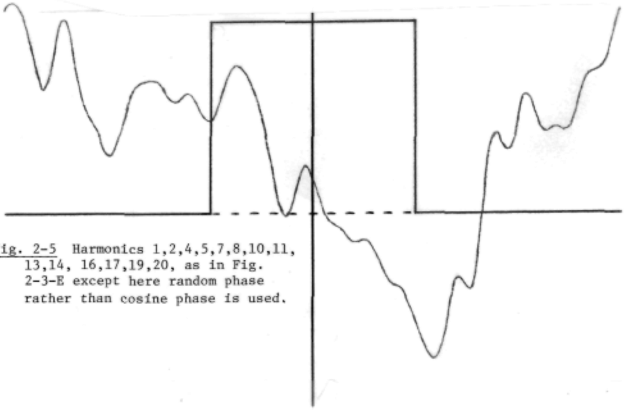


Fig. 2-5 Harmonics 1,2,4,5,7,8,10,11, 13,14, 16,17,19,20, as in Fig. 2-3-E except here random phase rather than cosine phase is used.

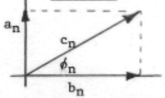
We have now determined the FS coefficients of the 1/3 duty cycle pulse, and have also determined the phase of the components as being all cosine phase for the positioning about zero that we chose. Consider why we got only cosine phase here. The pulse as we chose it is an even function $[f(-t) = f(t)]$. When we tried to get sine phase, the b_n coefficients, equation (2-3), we had to integrate this even f(t) as multiplied by the odd function $\sin(n\pi t/L)$. This product of an even function times an odd function gives an odd function, and in the integration from -L to 0, we gain a certain amount which is then cancelled out by the remaining part of the integration from 0 to +L. Thus there is an advantage to setting up the function f(t) with as much symmetry as possible (it is not always possible of course). If we had set up f(t) so that it was an odd function (not possible with the pulse), we would have obtained b_n coefficients, but no a_n. Now, if we had positioned the pulse within the interval -L to +L in an arbitrary manner, it would be neither even nor odd, but would be composed of an even part and an odd part. There would then be both a_n and b_n coefficients in the FS. The a_n coefficients would represent the cosine phase (the even part) while the b_n coefficients would represent the sine phase (the odd part) of the function f(t). Sine and Cosine are 90° out of phase. Thus when there is an a_n coefficient and a b_n coefficient for the same frequency (same n), what is really being said is that there is a single component with magnitude:

$$C_n = \sqrt{a_n^2 + b_n^2} \qquad (2\text{-}13)$$

(remember Pythagoras) and phase (see Fig. 2-6) given by:

$$\phi_n = \mathrm{Tan}^{-1}\,(a_n/b_n) \qquad (2\text{-}14)$$

Fig. 2-6



To get some ideas about the effect of the number of components in a FS and their relative phases, the reader may study the drawings in Fig. 2-3, Fig. 2-4, and Fig. 2-5 on the previous two pages. Fig. 2-3 shows the addition of the FS components of the 1/3 duty cycle pulse one at a time for the first four non-zero components, and then the sum of the first 14 non-zero components. For reference, the full 1/3 duty cycle pulse is also drawn. Thus it is possible to appreciate how each additional component improves the approximation. The importance of phase is shown in Fig. 2-4 and Fig. 2-5. Each of these has the same harmonics and in the same proportions as Fig. 2-3-E. In Fig. 2-4, instead of using the cosine phase required for the original pulse, sine phase is used. Note that the resulting waveform is in fact odd (ignoring the DC term, which is really an a term a_o) as we would expect from the sum of sine terms only, the sine function being odd. The resulting waveform certainly does not resemble the pulse. Fig. 2-5 shows the same components, but here an arbitrary or random phase has been assigned to each of the components. This waveform not only does not resemble the pulse, but has lost all symmetry.

We have thus seen that different phase choices for the components of a FS can result in extreme differences in the resulting time waveform. What we have suggested however, is that these phase differences make little or no difference to the ear. Each of the waveforms, Fig. 2-3-E, Fig. 2-4, and Fig. 2-5, have exactly the same amplitude spectrum, and they sound almost exactly the same to the ear. We can discuss this a bit. This "phase deaf" property of the ear is pretty well verified by experiment. Any ability we may have to detect phase is residual to some other process, and not "intentional." If we construct the waveforms of Fig. 2-3-E, Fig. 2-4, and Fig. 2-5, and play them through a loudspeaker into a room, it is doubtful that any listener would be able to hear any difference between them, let alone be able to consistently identify them. If instead we play them to a listener through headphones, the listener may be able to hear very small differences. From the headphone experiments, we can suggest that perhaps the fact that Fig. 2-4 has a larger peak amplitude, or some secondary effect of this larger peak, may allow us to tell it from the others. The results from playing into a room can also be understood. Inside the room the sound that actually reaches our ears consists of a direct sound and numerous reverberations. Thus any initial phase arrangement is greatly scrambled by the time it reaches our ear. Thus we would expect either great confusion (which would seem to have been removed by human evolution - when the sabre tooth tiger roared in the caves of our ancestors, the last thing they needed was confusion!), or the ear-brain must have evolved to make some consistent sense out of the mess. The latter seems to be the case, and the sense seems to be to just work with the amplitude spectrum.

While these points about the effects of relative phase are interesting, there is perhaps an even more important point that we can infer at this point. Suppose you were asked to predict which of the three waveforms, The 1/3 duty cycle pulse, the sine-phased waveform of Fig. 2-4, or the random-phased waveform of Fig. 2-5, would be the most musical (or let's suppose equivalently, the least electronic sounding). Well, from the above, you know the answer. They are all very much the same, and are very electronic sounding, each being a perfectly periodic waveform. [Keep in mind that each of the drawings of Fig. 2-3, Fig. 2-4, and Fig. 2-5 represent only one cycle of a periodic waveform.] If you have spent any time at all looking at musical waveforms from acoustic instruments on an oscilloscope, then you know that they look a lot more like Fig. 2-5 then they do like Fig. 2-4 or like the 1/3 duty cycle pulse. One this basis, you might have supposed that Fig. 2-5 would be more musical. Since this is not so, you are aware of the danger of trying to infer anything from the actual structure of the time waveform.

There are important points here. First, all periodic waveforms, regardless of the actual features in their time waveform, are electronic sounding. Thus there is little point in arranging circuitry or systems to control this detail to a large degree. Simpler easy to generate waveforms will do as well. Secondly, if we go back to the idea of observing waveforms from acoustic instruments, you probably recall your difficulty in obtaining a stable trace on the scope. You were perhaps thinking that if you could only get a stable trace, you could figure out how to duplicate the sound. Yet the whole story is being told by the scope. It is telling you that the waveform is not stable, but rather changing on a cycle by cycle basis. The scope is telling you ("warning you") not to try! If you succeed in isolating any one waveform cycle, and then play it back repeatedly, you will obtain only a periodic waveform, and regardless of its origins and details, it is going to sound electronic. The scope "knows" that the game can't be won. People perhaps have to learn the hard way.

This leads to the further point of trying to cause the time waveform to evolve on a cycle-by-cycle basis as a means of synthesis. This is possible, can be useful (perhaps more for animation of steady state than for full synthesis), and has been studied to a degree. Yet this scheme does ignore the fact that it is the spectrum that really matters, and by changing the details of the waveform, it is the spectrum that we are changing. Thus the waveform modification method may lead to the exact same result, but the control of the process is less obvious than it is in a more direct form of spectrum control.

2.1d LISTING OF WAVEFORMS AND THEIR FOURIER SERIES:

This part will be a listing of waveforms that are often encountered in musical engineering. Along with the name and shape, we will give the FS, and then make some comments on the timbre (tone color) of the waveform and its method of generation or likely occurrence.

1. Waveform: Sine Wave

   Shape: Fig. 2-7

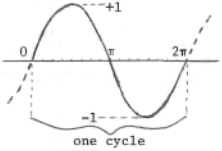


Fig. 2-7

Fourier Series: $f(t) = \sin t$ (2-15)

Harmonics and Timbre: Contains only one component, the fundamental. Timbre is very mellow. Virtually inaudible below about 100 Hz for normal amplitude levels. Thin and "flute-like" at high frequencies.

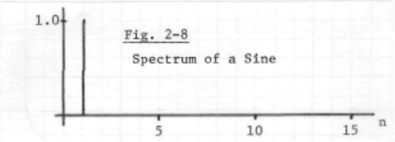


Fig. 2-8

Spectrum of a Sine

Origins: Sine wave generators are common for fixed frequencies. For variable frequencies (including VCO's), the sine function is usually obtained by shaping a triangle wave with a non-linear circuit, and perhaps using some filtering. Acceptable distortion levels for most music synthesizers are in the range of 1% to 2%.

2. Waveform: Sawtooth

Shape: Fig. 2-9

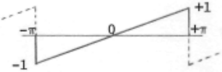


Fig. 2-9

Fourier Series:

$$f(t) = \frac{2}{\pi}\left[\sin(t) - (1/2)\sin(2t) + (1/3)\sin(3t) - (1/4)\sin(4t) + \ . \ . \ . \right] \qquad (2\text{-}16)$$

Harmonics and Timbre: All harmonics present, falling off as $1/n$. Timbre is generally "bright" and "full". May be "buzzy" at low frequencies, and "oboe-like" or "trumpet-like" at middle frequencies.

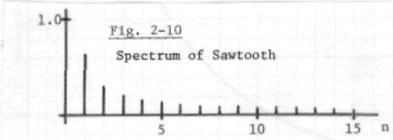


Fig. 2-10

Spectrum of Sawtooth

Origins: Sawtooth based VCO's are common, and thus the sawtooth is directly available. A common sort of generator is shown in Fig. 2-11 where a current source charges a capacitor. When the linear ramp thus produced exceeds a peak reference level, a comparator closes an electronic switch, discharging the capacitor to ground, thus starting another sawtooth cycle. The one-shot assures that the switch is closed long enough to discharge the capacitor sufficiently. Triangle-to-sawtooth methods are available for triangle based oscillators (see Triangle)

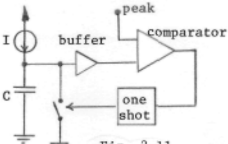


Fig. 2-11

3. Waveform: Square

Shape: Fig. 2-12

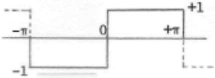


Fig. 2-12

Fourier Series:

$$f(t) = \frac{4}{\pi}[\ \mathrm{Sin}(t) + (1/3)\mathrm{Sin}(3t) + (1/5)\mathrm{Sin}(5t) + \ldots.] \qquad (2\text{-}17)$$

Harmonics and Timbre: All odd harmonics, falling off as $1/n$. The lack of even harmonics gives the waveform a "hollow" or "clarinet-like" sound over much of its range.

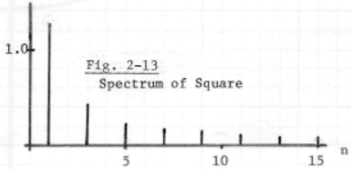


Fig. 2-13 Spectrum of Square

Origins: The square wave naturally occurs in a triangle based VCO as it is part of the generation process (see Triangle). When the square is not available directly, it is easily obtained by simply using a comparator on a triangle, sawtooth, sine, or other suitable waveform.

4. Waveform: Triangle

Shape: Fig. 2-14

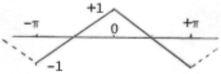


Fig. 2-14

Fourier Series:

$$f(t) = \frac{8}{\pi^2}[\ \mathrm{Cos}(t) + (1/9)\mathrm{Cos}(3t) + (1/25)\mathrm{Cos}(5t) + \ldots\ldots] \qquad (2\text{-}18)$$

Harmonics and Timbre: Contains (like the square) all odd harmonics, but here they fall off as $1/n^2$ instead of as $1/n$ with the square. This more rapid fall-off means that there is very little actual harmonic content present. Timbre is very mellow, much like the sine except at low frequencies where the triangle's harmonics make it more audible.

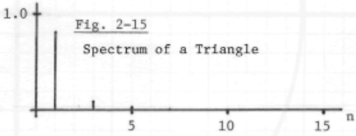


Fig. 2-15 Spectrum of a Triangle

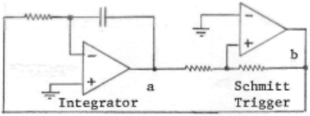


Fig. 2-16 Most basic form of a "Triangle-Square" or "Integrator-Schmitt-Trigger" oscillator

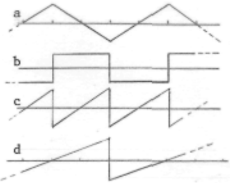


Fig. 2-17 Waveforms in a triangle based oscillator

Origins: In a sawtooth based VCO, the triangle can be obtained by full-wave rectification followed by a level shift of -1 as a sketch will show the reader. Another (the other) popular VCO design method is the triangle based design. This design is basically that of an integrator and a Schmitt trigger in a loop (Fig. 2-16).

In Fig. 2-16, the output of the Schmitt trigger (b) is either at its positive limit or at its negative limit. This causes the negative integrator to ramp in response. When the ramp (a) reaches the upper or lower threshold of the Schmitt trigger, the Schmitt trigger reverses, going to its other limit, and thus causing the ramp to turn around an go the other way. A triangle is thus produced, and the phase relationship between the triangle and the square is as shown in Fig. 2-17. Because of this relationship, it is possible to form the double frequency sawtooth (c) by using the square to invert (or not) the waveform (a). For example, when the square is low, we will not have our inverter working, but when the square is high, it will invert. Summing the square and the double-frequency sawtooth and dividing by two gives a sawtooth at the original frequency (d).

5. Waveform: Pulse

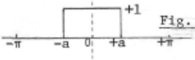


Fig. 2-18

Shape: Fig. 2-18

Fourier Series:

$$f(t) = \frac{a}{\pi} + (2/\pi) \sum_{n=1}^{\infty} \frac{\mathrm{Sin}(na)}{n} \mathrm{Cos}(nt) \qquad (2\text{-}19)$$

Harmonics and Timbre: The harmonics of the pulse depend on the parameter "a" where a/π is the so-called duty cycle (time high divided by total time for one cycle). The parameter a can be voltage-controlled resulting in pulse-width modulation. The term immediately following the summation sign is the most important for determining the spectrum. It tells us first that the harmonics fall off as 1/n, but superimposed on this fall off is the Sin(na) term. *When the duty cycle is a rational fraction, na is a multiple of* π and Sin(na) is zero. Weak regions in the spectrum are centered around values of n where na is near a multiple of π. In general the spectra are rich and bright, containing many significant harmonics.

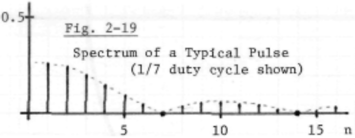


Fig. 2-19

Spectrum of a Typical Pulse (1/7 duty cycle shown)

Origins: The pulse is usually derived from another waveform by simply using a comparator (see Fig. 2-20). In a static condition (constant a), it makes

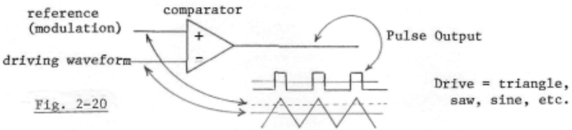


Fig. 2-20

no difference what the driving waveform is as long as a is set properly. In the dynamic condition (Pulse Width Modulation), it does make a difference, as not only will the driving waveform determine how a changes with changing reference level, but it will determine how the different edges of the pulse will move, resulting in significant and audible differences.

NOTE: The above five waveforms are the standard ones for most voltage-controlled synthesizers. The ones below are less common, but can be implemented on home-built equipment.

6. Waveform: Equal Spaced Double Pulse

Shape: Fig. 2-21

Fourier Series:

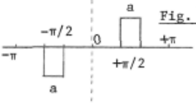


Fig. 2-21

$$f(t) = \frac{4}{\pi}[\ \mathrm{Sin}(a/2)\mathrm{Sin}(t) - (1/3)\mathrm{Sin}(3a/2)\mathrm{Sin}(3t) + (1/5)\mathrm{Sin}(5a/2)\mathrm{Sin}(5t) - \ldots\ ] \qquad (2\text{-}20)$$

Harmonics and Timbre: Spectrum consists of odd harmonics only. The Sin(na/2) term serves to place a spectral envelope over these components similar to that of the regular pulse (see Fig. 2-19). The example spectrum below is for $a = \pi/4$. It is thus missing every 8th harmonic from the Sin(na/2) term. These terms are also missing because they are odd, so the actual null at n=8, 16, 24, etc. is an indication of a null in the spectral envelope. Components also go as 1/n. Timbre is quite hollow except for very small a.

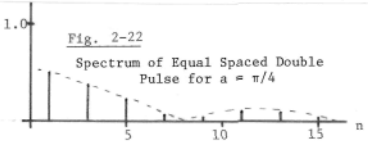


Fig. 2-22

Spectrum of Equal Spaced Double Pulse for $a = \pi/4$

Origins: The equal spaced double pulse is generated in a manner similar to that of Fig. 2-20 for the regular pulse, except a second comparator, referenced to a negative version of the original reference, and driven by the same driving waveform, is added. The sum of the original pulse and the pulse from the added comparator are summed (and scaled as needed).

7. Waveform: Common Edge Double Pulse

Shape: Fig. 2-23

Fourier Series:

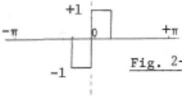


Fig. 2-23

$$f(t) = \frac{4}{\pi}[\ \mathrm{Sin}^2(a/2)\mathrm{Sin}(t) + (1/2)\mathrm{Sin}^2(2a/2)\mathrm{Sin}(2t) + (1/3)\mathrm{Sin}^2(3a/2)\mathrm{Sin}(3t) + \ldots\ ] \qquad (2\text{-}21)$$

Harmonics and Timbre: Here the spectrum contains all harmonics, falling off as 1/n, and at the same time, there is a $\mathrm{Sin}^2(na/2)$ term multiplying each of these. Thus there may be missing components if na/2 is a multiple

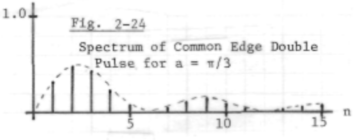


Fig. 2-24

Spectrum of Common Edge Double Pulse for $a = \pi/3$

of π, and weak regions of the spectra appear where $na/2$ is close to a multiple of π. In the example spectrum of Fig. 2-24, a is $\pi/3$ so $na/2 = n\pi/6$, so we have nulls for each 6th harmonic, as can be seen. Note however that the spectrum here is different from that of the pulse in that we have a decreasing spectral envelope as n=0 is approached (compare Fig. 2-19 to Fig. 2-24). The timbre of the common edge double pulse is "pulse-like" except the fundamental is weak, particularly for smaller values of a.

These spectra for the two double pulses thus supplement the regular pulse in that the equal spaced version supplies a pulse-like spectral envelope with only odd harmonics, while the common edge version supplies a pulse-like spectral density that decreases for low harmonics. Note that these pulse spectra can be obtained as the sum of spectra of the two pulse components taken individually. Thus frequency components that would be missing from individual pulses are missing in the sum. Study will show that the basic pulse duty cycles for Fig. 2-19, 2-22, and 2-24 are 1/7, 1/8, and 1/6 respectively. Thus we are able to understand the missing 7th, 8th, and 6th harmonics respectively. The other alterations of the shape of the spectra are the result of components of the same frequency as generated by the separate pulse components interfering with each other.

Origins: The common edge double pulse is simply generated by the same method as the equal spaced version, except for the common edge, the driving waveform is a sawtooth instead of a triangle or sine. Fig. 2-25 shows the way this comes about.

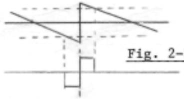


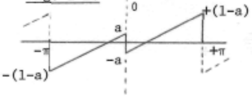
Fig. 2-25

8. Waveform: Symmetrized Ramp

Shape: Fig. 2-26

Fourier Series:

Fig. 2-26



$$f(t) = \frac{1}{\pi}[\ 2(1-2a)\mathrm{Sin}(t) - \mathrm{Sin}(2t)$$
$$+ (2/3)(1-2a)\mathrm{Sin}(3t) - (1/2)\mathrm{Sin}(4t)$$
$$+ (2/5)(1-2a)\mathrm{Sin}(5t) - (1/3)\mathrm{Sin}(6t)$$
$$+ (2/7)(1-2a)\mathrm{Sin}(7t) - (1/4)\mathrm{Sin}(8t) + \ldots\ ] \qquad (2\text{-}22)$$

Harmonics and Timbre: The symmetrized ramp has two different sets of harmonics, the even ones which are a fixed set corresponding to a double frequency sawtooth, and a set of odd harmonics whose amplitude depends on the value of the parameter a. The even harmonics, being sawtooth, fall off as 1/n. The odd harmonics also have a 1/n dependence in addition to their dependence on the parameter a. The timbre of the symmetrized ramp is quite similar to that of the sawtooth over much of the range of a, but in the range of a=0.3 to 0.4, where the second harmonic is somewhat stronger than the fundamental, the timbre has a strong octave cue.

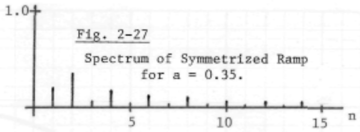


Fig. 2-27

Spectrum of Symmetrized Ramp for a = 0.35.

Origins: Generation of a symmetrized ramp can be accomplished by adding an appropriate portion of an appropriately phased square wave to the sawtooth. The usual way this can come about is through the process of triangle-to-sawtooth conversion suggested in Fig. 2-17. If the mix is right, we get a sawtooth, and if not, it is easy to get the symmetrized ramp, a useful byproduct.

9. Waveform: Pi-Sine (Sawtooth Driven Sine Shaper)

Shape: Fig. 2-28

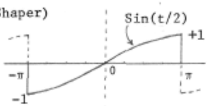


Fig. 2-28

Fourier Series:

$$f(t) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{8n}{4n^2 - 1}\,\text{Sin}(nt) \qquad (2\text{-}23)$$

Harmonics and Timbre: The waveform contains all harmonics, and is similar to a sawtooth except it has less harmonic content. For large values of n, the harmonics fall off approximately as 1/n. The timbre is bright and rich, but less harsh than the sawtooth.

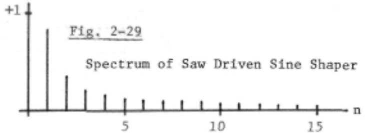


Fig. 2-29

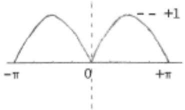
Spectrum of Saw Driven Sine Shaper

Origins: The Pi-Sine is composed of half a sine wave, and thus one π of the possible two π of the sine waveform. It is easily generated by extending the usual triangle-to-sine conversion process. The triangle to sine converter consists of a non-linear circuit that bends over slightly for higher voltages. This has the effect of rounding the peak of the triangle to a sine-like waveform. Many non-linear circuits can be used, including certain characteristics of FET's and of BJT differential amplifier input stages. The generation of the Pi-Sine is just a matter of driving this converter with a sawtooth wave instead of with the triangle.

10. Waveform: Full-Wave Rectified Sine

Shape: Fig. 2-30

Fig. 2-30



Fourier Series:

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\text{Cos}(2nt)}{4n^2-1} \qquad (2\text{-}24)$$

Harmonics and Timbre: The spectrum contains only even harmonics, falling off as $1/n^2$ for large n. Because the waveform has no fundamental (n=1), it would sound like a waveform with pitch corresponding to 2n containing all harmonics. [That is, a waveform with only even harmonics makes no sense]. The spectrum is shown in Fig. 2-31, neglecting the DC term. Timbre is somewhat mellow, although there is enough harmonic content to add some interest.

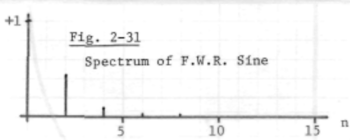


Fig. 2-31

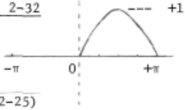
Spectrum of F.W.R. Sine

Origins: The FWR sine can be obtained with an ordinary FWR or absolute value circuit. Note that it is the function assumed to be the input of many power supply filters, although this is not always what the actual waveform looks like.

11. Waveform: Half-Wave Rectified Sine

Shape: Fig. 2-32

Fig. 2-32



Fourier Series:

$$f(t) = \frac{1}{\pi} + \frac{\mathrm{Sin}(t)}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\mathrm{Cos}(2nt)}{4n^2-1} \qquad (2\text{-}25)$$

Harmonics and Timbre: The spectrum of the HWR sine contains all even harmonics, and also contains a fundamental, so it, unlike the FWR sine, maintains the pitch of the fundamental (n). Note that the even harmonics have the relative amplitudes of the FWR sine case, but are only half *as strong. The HWR sine has slightly more perceived harmonic content* then the FWR sine (with both at the same pitch). This is the first waveform considered that consists of a fundamental and then even rather than odd harmonics. There is somewhat of a feeling of a hollow space between the fundamental and its harmonics, perhaps because the even harmonics group to form what is really an octave. [All second harmonics are octaves of course, but without odd harmonics interleaving the even ones, reference back to the true fundamental may not be so solid.]

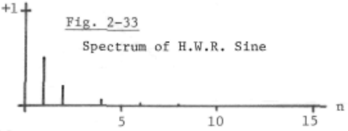


Fig. 2-33

Spectrum of H.W.R. Sine

Origins: The HWR sine is obtained by a half-wave rectifier which may be a precision source with an op-amp or two, or may be obtained with a simple resistor and diode.

12. Waveform: Double Clipped Sine Wave

Shape: Fig. 2-34

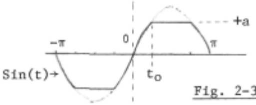


Fig. 2-34

Fourier Series:

$$f(t) = \frac{2}{\pi} \sum_{n=1,3,5..}^{\infty} \left(\left[\frac{\mathrm{Sin}(1-n)t_o}{(1-n)} - \frac{\mathrm{Sin}(1+n)t_o}{(1+n)} \right] + \frac{4\,\mathrm{Sin}(t_o)}{n\pi} \left[\mathrm{Cos}(nt_o) - \mathrm{Cos}(n\pi/2) \right] \right)^{*} \qquad (2\text{-}26)$$

Harmonics and Timbre: The exact amount of harmonic content, here properly referred to as "harmonic distortion" (or a sine wave) will depend on the clipping point $a = \mathrm{Sin}(t_o)$. Only odd harmonics are present however. A typical spectrum, for $t_o = \pi/4$ is shown in Fig. 2-35. As $t_o \to 0$, the spectrum becomes that of a square wave.

Origins: *Clipping on both top and bottom occurs typically when the* input signal to an amplifier is too large, and the amplifier clips

*treat the n=1 case in a manner similar to equation 2-12 at the power supply levels. Intentional uniform clipping for the purpose of generating harmonics can be done with a clipping circuit, typically implemented with two series back-to-back zener diodes in parallel with a feedback resistor in an op-amp inverter.

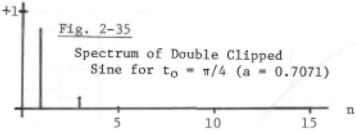


Fig. 2-35

Spectrum of Double Clipped Sine for $t_o = \pi/4$ ($a = 0.7071$)

13. Waveform: *Top Clipped Sine Wave*

Shape: Fig. 2-36

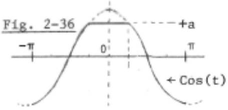


Fig. 2-36

Fourier Series:

$$f(t) = (1/\pi)[t_o \mathrm{Cos}(t_o) - \mathrm{Sin}(t_o)]$$
$$+ (1/\pi)[\pi - t_o - \tfrac{1}{2}\mathrm{Sin}(2t_o) + 2\,\mathrm{Cos}(t_o)\mathrm{Sin}(t_o)]\,\mathrm{Cos}(t)$$
$$+ (1/\pi) \sum_{n=2}^{\infty} \left[\frac{2\,\mathrm{Cos}(t_o)}{n}\mathrm{Sin}(t_o) - \frac{\mathrm{Sin}(1-n)t_o}{(1-n)} - \frac{\mathrm{Sin}(1+n)t_o}{(1+n)} \right] \mathrm{Cos}(nt) \qquad (2\text{-}29)$$

Harmonics and Timbre: The harmonic content and the resulting timbre depend on the clipping level $a = \mathrm{Cos}(t_o)$. This one sided clipping results in the generation of even harmonics as well as the odd ones you get with two-sided clipping. A typical spectrum, for $t_o = \pi/2$, is shown in Fig. 2-37. Because of certain difficulties due to limits on the terms of the Fourier Series for $n = 0$ and $n = 1$, these first two terms were written out individually above. Fig. 2-37 neglects the DC term.

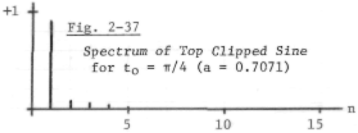


Fig. 2-37

Spectrum of Top Clipped Sine for $t_o = \pi/4$ ($a = 0.7071$)

Origins: Clipping on one side only is not uncommon, even against equal power supply levels, since there is often a DC offset involved with high gain conditions. Circuits for intentional clipping are also possible, again using zener diodes in feedback loops.

Obviously this game of listing waveforms and their spectral content could go on much longer, but here we will stop since we have covered the major cases. Other cases can be worked out as needed. The reader will note however the complexity of actually writing down some of the later series on the list. While we have found a good variety of spectra, we should not forget one thing. All of these, being perfectly periodic, are all roughly equally boring, and compared to the wide variety of sounds available to human ears, there is relatively little variety here. Thus we will need to look at ways of creating dynamic spectra. We will go on a bit more to some other aspects of Fourier analysis and related transform methods, and then move on to filtering. Some additional materials on Fourier Series are found in Appendix A.