## STEP RESPONSE WITH FEEDBACK – LAPLACE VIEW

This webnote expands on the previous webnote [1] which in turn expanded on EN#219 [2]. The changes here are that we add a gain factor k to the network (Fig. 1), and we do the step function by Inverse Laplace Transform. Here k can be anything, but we have in mind to use it mainly as k=+1 or k=-1 according to whether or not we want to look at positive or negative feedback. At the same time, we are retaining the actual voltage-divider/capacitor for F(s) rather than subsuming it into a time constant, thinking that this will be more intuitive. The general value of A is used (rather than just A=1) since we want to use a voltage-divider attenuator (strictly positive between 0 and +1) and still investigate loop gains through +1.
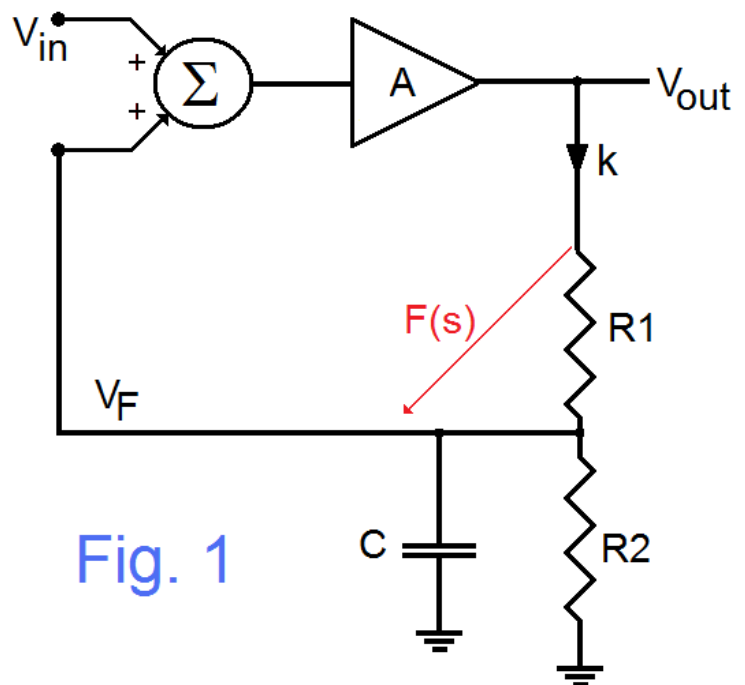


Fig. 1

Finding the transfer function of Fig. 1 is not difficult, and here we can even just modify slightly the previous result [1] by replacing A with Ak and then "backing up" to the amplifier output by dividing the whole result by k. That is:

$$T(s) = A \, \frac{(R_1+R_2)+ sCR_1R_2}{R_1+(1-Ak)R_2+sCR_1R_2} \tag{1}$$

which can be written in a form:

$$T(s) = A \; \frac{\left(\frac{1}{CR_1} + \frac{1}{CR_2}\right) + s}{\left[\frac{1}{CR_2} + (1 - Ak)\frac{1}{CR_1}\right] + s} \tag{2a}$$

$$= A \frac{(s+\alpha)}{(s-\beta)} \tag{2b}$$

where:

$$\alpha = \left(\frac{1}{CR_1} + \frac{1}{CR_2}\right) \tag{3a}$$

is the negative of the zero and:

$$\beta = \left[\frac{1}{CR_2} + (1 - Ak)\frac{1}{CR_1}\right] \tag{3b}$$

is the pole.  This we prepare for inverse Laplace transform by writing as two terms:

$$T(s) = A\left[\frac{s}{s-\beta} + \frac{\alpha}{s-\beta}\right] \tag{4}$$

So the Laplace transform of the step response is T(s)/s or:

$$\frac{T(s)}{s} = A\left[\frac{1}{s-\beta} + \frac{\alpha}{s(s-\beta)}\right] \tag{5}$$

which is very easy to invert using the two tabulated pairs (6a) and (6b):

$$\frac{1}{s-\beta} \leftrightarrow e^{\beta t} \tag{6a}$$

$$\frac{1}{s(s-\beta)} \leftrightarrow \frac{e^{\beta t} - 1}{\beta} \tag{6b}$$

Thus the step response of (5) is:

$$s(t) = A\left[e^{\beta t} + \alpha\frac{e^{\beta t}-1}{\beta}\right] \tag{7}$$

which is an easily calculable equation.

Fig. 2 (negative and zero feedbacks) and Fig. 3 (positive feedbacks) show a range of 12 examples.  There the underlined loop-gain g is calculated as g = AkF(0), where F(0) = F(s=0) is the

attenuation of the feedback voltage divider: $F(0) = R_2/(R_1+R_2)$.  In all our examples we have taken A=2 (A is the <u>open-loop-gain</u> – no gain or attenuation because the feedback is blocked) and the values of $R_1$ and $R_2$ set the feedback divider to F(0) as noted.  By use of the capacitor C we get to see the step response actually change from an initial value A = 2 to a value of $G_c = A \times [\frac{1}{1-g}]$ on the right side [1].  Here $G_c$ is the <u>closed-loop-gain</u>.  (The R-C time constants are set experimentally to better show the convergence.)   If you are unclear about the various "loop-gains" please see Fig. 6 of the Appendix.  The open-loop-gain is on the left at time 0 (always A=2 in Fig. 2 and Fig. 3), and the response moves with time to the
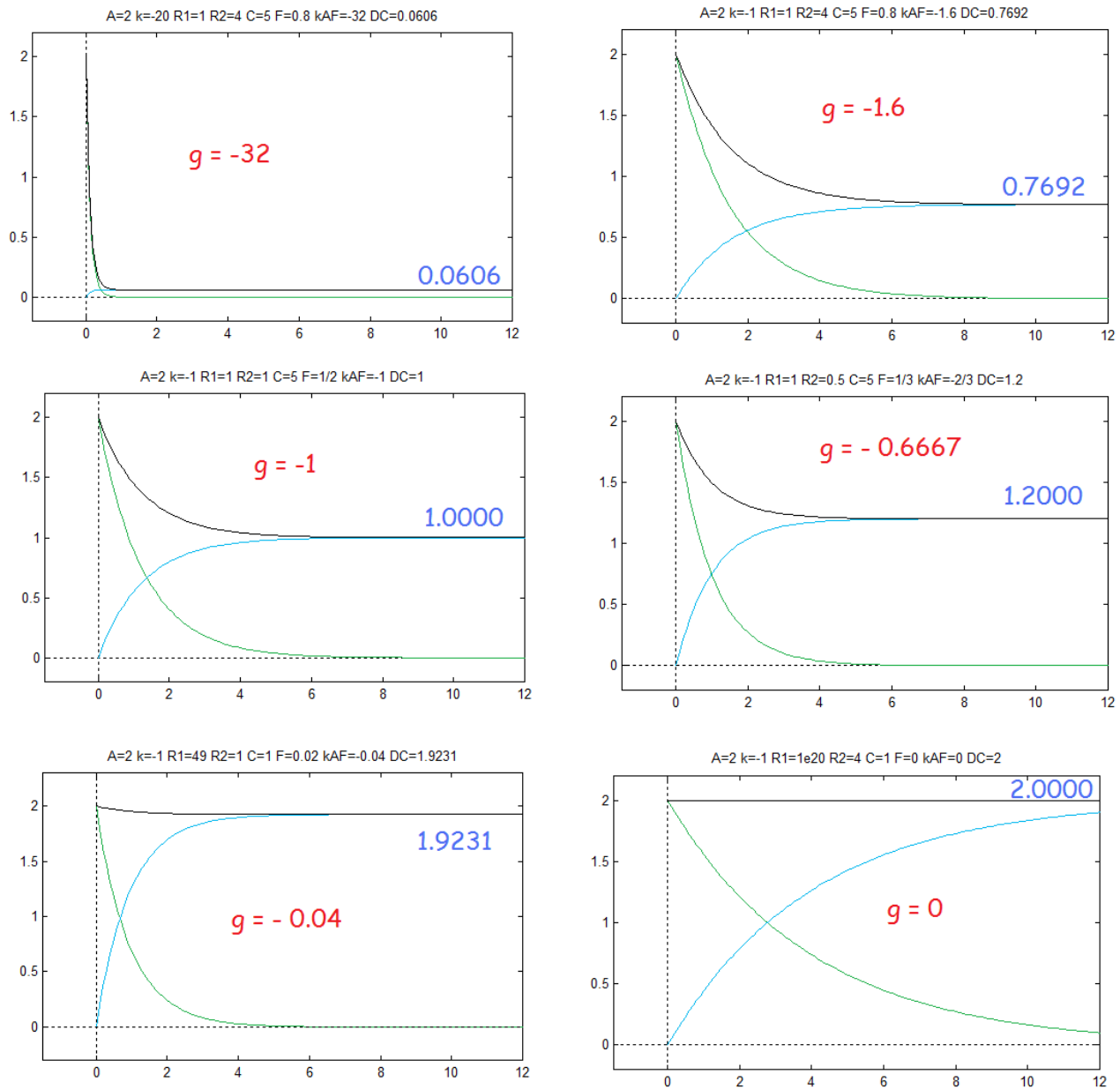


**Negative and Zero Feedback**

**Fig. 2**

closed-loop-gain, asymptotically on the right at time 12. Each of the graphs also shows the green high-pass component [first term of equation (7)] and the blue low-pass component [second term of equation (7)]. Here g begins at g=-32 (upper left of Fig. 2) which is large, and which we achieved here by setting k = -20. (Note that many op-amp circuits have far larger negative feedbacks.) As we increase g, the values at the end of the step response (the DC gain) increases. Eventually the loop gain g approaches 1 (lower panels of Fig. 3), the limit of stability.
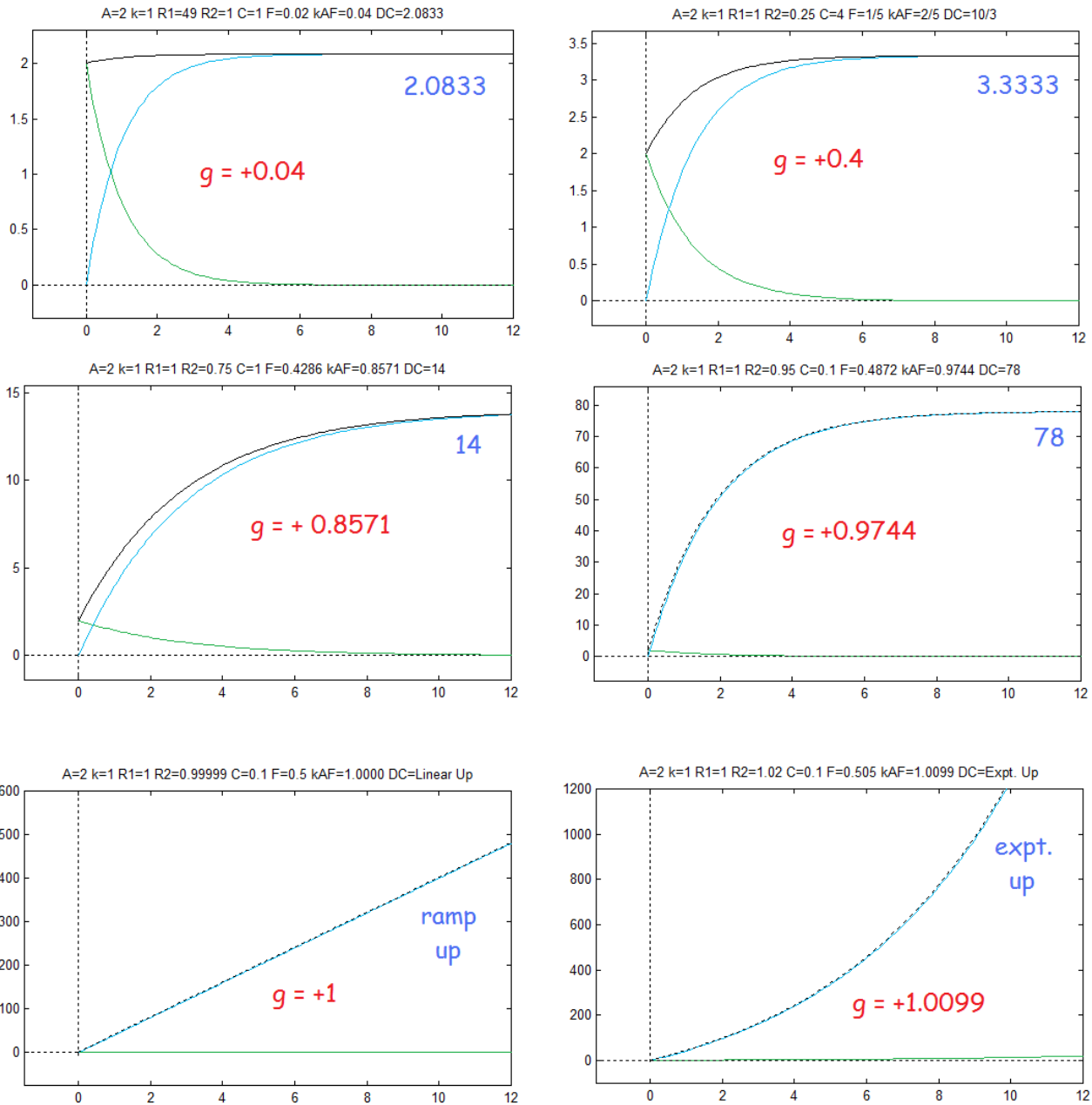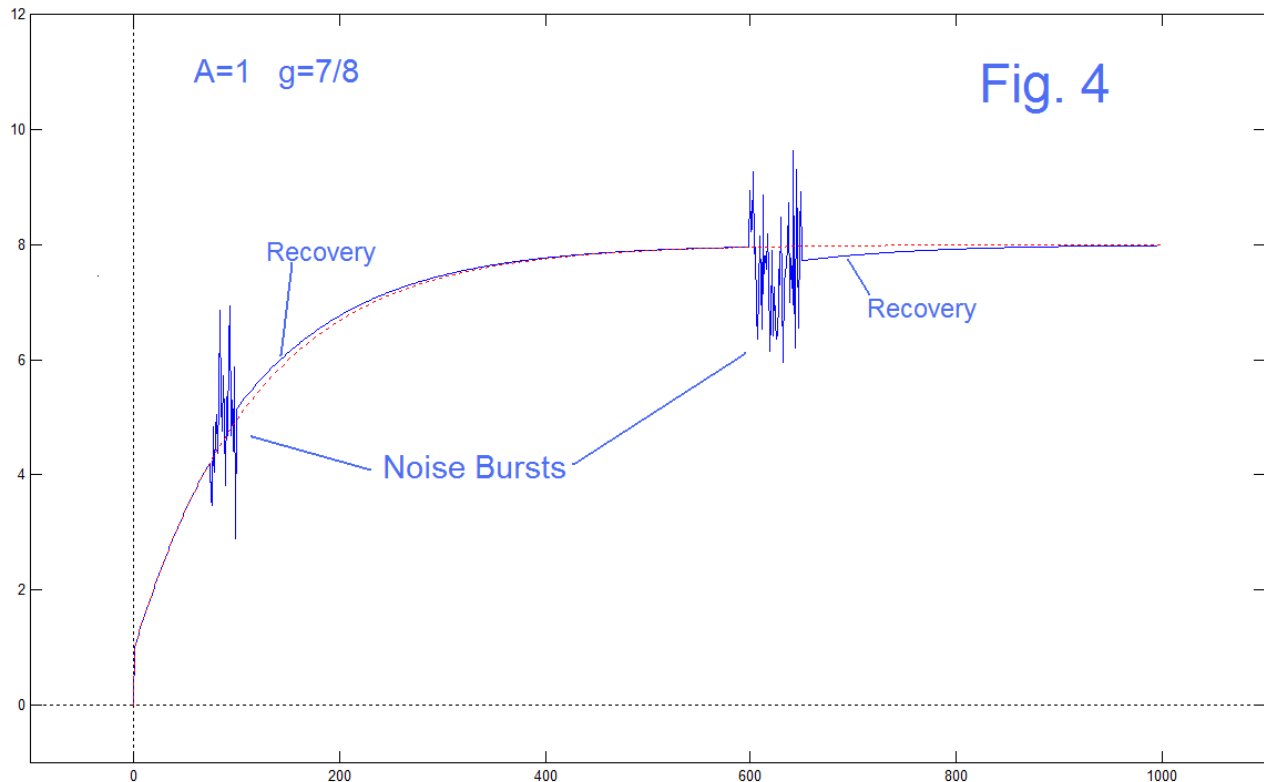


Positive Feedbacks                    Fig. 3

From among the 12 examples of Fig. 2 and Fig. 3, we have noted g = -32 as the largest negative feedback, with a reduction of $G_c$ to 0.606 down from A=2. On the other extreme, as g approaches (and exceeds 1) we have g=0.9744 with its closed loop gain of 78, g=1 which is an unstable ramp up, and g=1.0099 which is an exponential ramp up. The range from g=-∞ to just short of g=+1 is stable. Keep in mind that, relative to g=+1, gains such as g=0.8 are more negative. See Fig. 5 of the appendix.

To make the point about gains being close to zero, but positive or negative, see g=-0.04, g=0, and g=+0.04. With g=0 (no feedback) $G_c$ is of course just A=2. Note that the gains just below and just above g=0 are similar and modest. Nothing special happens at g=0. Nothing blows up with positive feedback until g=1 or larger.

This does not mean that we want to use very large positive feedbacks. Here we have in the loop a resistor divider. For ordinary resistors, real resistors with tolerances, the actual g might be well away from g=0.9 in any particular realization. This is a very sensitive case, and is not reliable. None the less, it can often be trimmed as with the Deliyannis band-pass [1]. One might in addition wonder if perhaps noise could trigger an instability.

Fig. 4 shows a case where A=1, g=7/8, and the step response heads for 1/(1-7/8) = 8. Here we show added to the step input two bursts of noise. Note that the noise more or less goes right through (and the response exceeds 8) but the feedback loop recovers (the blue response heads back to the red-dashed noise-free step response).



A=1  g=7/8          Fig. 4

Recovery

Recovery

Noise Bursts

## REFERENCES

[1]  B. Hutchins, "A Frequency-Sensitive Feedback Path Example,"  Electronotes Webnote ENWN-25,  3/30/2015
http://electronotes.netfirms.com/ENWN25.pdf

[2]  B. Hutchins, "Feedback Revisited – Gain Due To Feedback",  ***Electronotes***,  Volume 23, Number 219, November 2013
http://electronotes.netfirms.com/EN219.pdf

## Appendix

Here Fig. 5 shows the famous feedback curve [2] for the case of A=2 and for eight of the values of g from the 12 examples of Fig. 2 and Fig. 3.  That is, $G_c = A/(1-g)$.  The other four examples would be outside the range plotted.
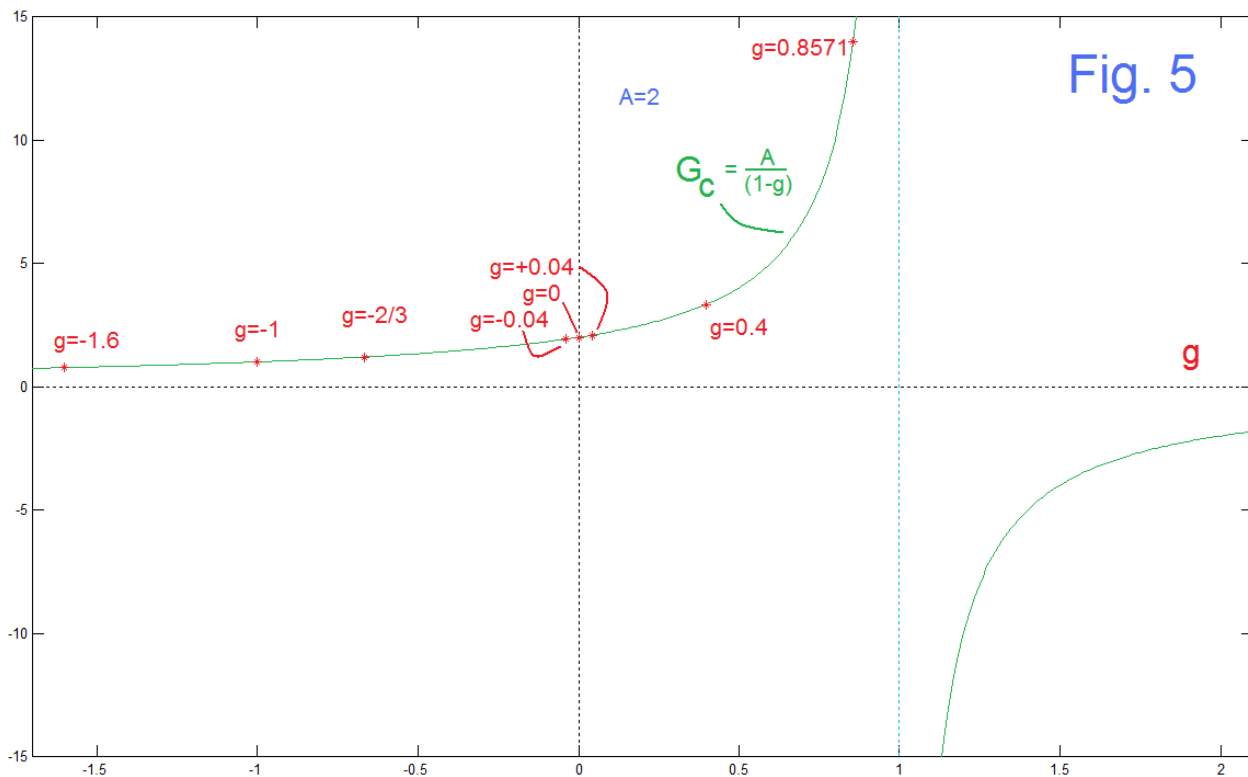
Fig. 6 is provided as a reminder for those who are unclear about the differences in the three types of "loop-gain". The open-loop-gain and closed-loop-gain are exactly what they say. The term "loop-gain" is the product of the gains around the feedback loop. At times it is said you get the loop-gain by breaking the loop (wherever you want) and calculating the gains from the break back around to the break. True enough, but why? You might forget to reconnect it!
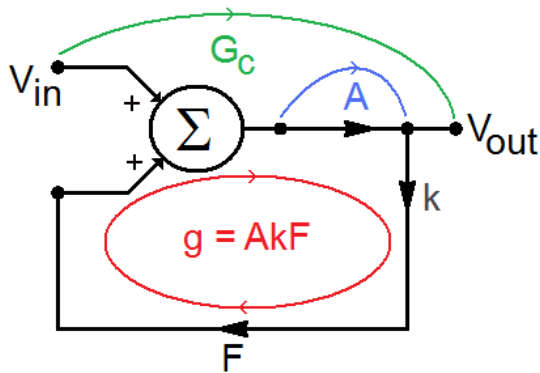


A = Open Loop Gain

Fig. 6

$g = AkF$ = Loop Gain

$G_C = A / [1 - g]$
= Closed Loop Gain