The "Four-Pole Low-Pass VCF", the cascade of four first-order low-pass sections with an overall positive feedback loop around the cascade was the brainchild of Bob Moog, based mainly on his engineering intuition, based on regenerative radio receiver design (feedback can sharpen and amplify – who remembers the "tickler coil"?). Bob did not know exactly where the poles moved when feedback was used. (When I found them and told him I had found them, he asked where they were.) My method of finding the poles (solving a 4^{th}-order polynomial, a non-trivial task at that date), used first in EN#85, was in retrospect, both clever and stupid. Richard Bjorkman, in EN#97 pointed out the really smart way of solving the problem (a simple change of variable!). This provided an analytical solution (and hence, insight) to a problem which in the mean time had yielded to numerical methods. Richard was kind enough to characterize my derivations as "unnecessarily cumbersome" – a term I soon came to love and use myself.

In posting EN#85, I thought it essential to also post Richard's "Brief Note", and this follows below:

Bernie, July 2011

A Brief Note On Polygon Filters:

-by Richard Bjorkman

In EN#95 (3), Bernie Hutchins describes the migration of poles as a function of feedback in what he terms "polygon" filters. While Bernie's results will certainly be of interest to anyone designing voltage-controlled low-pass filters, his derivation of these results seems to be unnecessarily cumbersome. In this article, we will outline a more direct derivation.

Briefly reviewing Bernie's article, we recall that an n-th order polygon filter is formed by cascading n identical first-order low-pass sections and then feeding the output of the last section back to the input of the first section with a feedback gain g. If each section is normalized (i.e. if RC = 1 for each section), then the poles of the filter are given by:

$$(s+1)^n - g = 0 \tag{1}$$

The solutions to equation (1) *are found to lie at the corners of an n-sided regular* polygon with center at -1 in the complex plane. The size of the polygon and its angular orientation in the complex plane are both found to be functions of g.

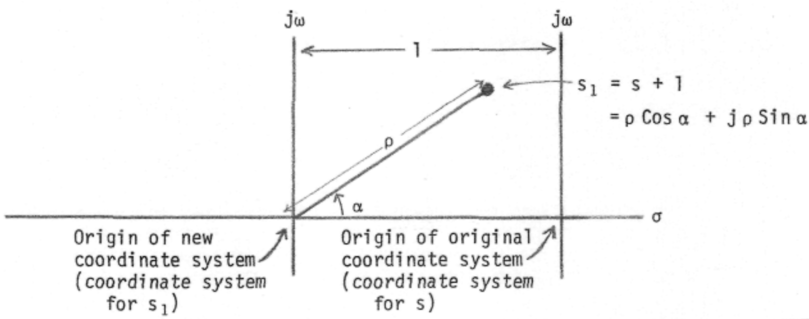
Bernie's method of solving equation (1) was to expand $(s+1)^n$ into a polynomial and then solve the resulting polynomial equation. For $n > 2$, this process is generally quite difficult. In a postscript to the article, however, Bernie notes: "If we look at the poles we found with reference to -1 in the complex plane, they look like the n-th roots of a complex number, with a possible overall rotation added on . . . . This all raises the possibility of working backward or forward or both to arrive at a formulation that does not involve the elplicit solution to the n-th order equations." It turns out that Bernie is on the right track here. In fact, with a suitable coordinate system, these poles can be viewed as the complex n-th roots of g itself, and the unwieldy polynomial equations can be completely eliminated.

The basic trick is to introduce a new variable s_1 by:

$$s_1 = (s+1) \tag{2}$$

Equation (2) can be thought of as establishing a new coordinate system whose axes are parallel to the axes of the original coordinate system, but whose origin lies at -1 on the original coordinate system (see figure below); s_1 is related to the new coordinate system in the same way that s is related to the original coordinate system. We observe that the origin of the new coordinate system lies at the center of the polygons that Bernie found. And, if we substitute equation (2) into equation (1), we find that equation (1) becomes simply:

$$s_1{}^n = g \tag{3}$$



It is useful at this point to impose polar coordinates on the <u>new</u> coordinate system. To convert s_1 to polar coordinates, we can define a non-negative radius ρ and an angle α (see illustration above) such that:

$$s_1 = \rho \cos\alpha + j\rho \sin\alpha \tag{4}$$

or, by Euler's formula:

$$s_1 = \rho e^{j\alpha} \tag{5}$$

Similarly, we can convert g to polar coordinates by writing:

$$g = \begin{cases} g\, e^{j\cdot 0^\circ} & \text{for } g \geq 0 \\ -g\, e^{j\cdot 180^\circ} & \text{for } g < 0 \end{cases} \tag{6}$$

We use two formulations for equation (6) to insure that the radius (g for $g \geq 0$, $-g$ for $g < 0$) is always non-negative.

We now substitute equations (5) and (6) into equation (3):

$$\rho^n e^{jn\alpha} = \begin{cases} g e^{j\cdot 0^\circ} & \text{for } g \geq 0 \\ -g e^{j\cdot 180^\circ} & \text{for } g < 0 \end{cases} \tag{7}$$

We obtain:

$$\rho = |g|^{1/n} \quad \text{for all } g \tag{8}$$

and:

$$\alpha = \begin{cases} \dfrac{k\cdot 360^\circ}{n} & \text{for } g \geq 0 \\ \dfrac{180^\circ}{n} + \dfrac{k\cdot 360^\circ}{n} & \text{for } g < 0 \end{cases} \tag{9}$$

$$\text{where } k = 0, 1, 2, \cdots, (n-1)$$

Equations (8) and (9) enable us to find the radius ρ and n distinct angles α. These solutions for ρ and α determine the locations of the n poles. To find the locations of the poles with reference to the <u>original</u> coordinate system, we simply combine equations (2) and (4) to obtain:

$$s = -1 + \rho \operatorname{Cos} \alpha + j \rho \operatorname{Sin} \alpha \tag{10}$$

The reader can easily verify that our results, as embodied in equations (8), (9), and (10), are completely consistent with Bernie's findings, and that the poles really do lie at the corners of an n-sided regular polygon in the manner that Bernie describes. One point is particularly worth noticing: from the two formulations of equation (9), we see that, as g passes from positive values to negative values, the polygon undergoes a rotation whose angle is given by:

$$\theta = 180^\circ/n \tag{11}$$

This rotation, whose exact source is something of a mystery in Bernie's article, thus turns out to be simply a natural consequence of the sign of g.

* * * * * * * * * * * * * * * * *