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GROUP ANNOUNCEMENTS

Once again I have been away from producing new material too long. I have been busy on another project. The article below I started probably 3 years ago, and I have expanded it here, and it was clearly a newsletter item rather than an application note.

NEGATIVE FREQUENCIES – AND THROUGH-ZERO FREQUENCY MODULATION

-by Bernie Hutchins

INTRODUCTION:

The technique of Frequency Modulation (FM) as a means of generating complex audio spectra for musical sounds has a long history going back to Moog in 1965 [1], and was advanced in a digital context by Chowning [2] in 1973. We have carried many items on FM in ***Electronotes*** including a major reviews [3]. Implementation of FM in an analog context (with exponentially controlled voltage-controlled oscillators) (VCOs) involved special considerations [4] in the case where the depth of modulation was time-varying, as was almost essential for the most useful effects.

It soon became apparent that for the really great effects, that not only dynamic depth (a time-varying modulation index) , but also a modulation range through zero frequency was very very nice to have. A “direct digital” (just calculate the samples) was always possible, as was a simple application of a frequency shifter [5]. However, the frequency shifter was a large hardware investment – compared to a VCO. The heterodyning principle of the frequency shifter approach led to a through-zero VCO by Jan Hall [6] and eventually led to a really neat digital oscillator approach [3] – one of those rare cases in engineering where every change necessary seems to help the overall design.

In between, a fairly simple "through-zero" VCO (TZVCO) appeared that did not require an analog multiplier (expensive), first suggested by Douglas Krahl I believe [7] and later added as a standard design [8]. This was based on a time-reversal interpretation of a negative frequency, which I believe, is both useful and 100% valid.

Here we shall make some comments on through-zero FM, not particularly as a review, but as a summary.

WHAT THE HECK ARE NEGATIVE FREQUENCIES?

The concept of through-zero frequency modulation (TZFM) is perplexing; first of all, because it involves the concept of negative frequencies. The actual notion of a negative frequency provides no problem mathematically. Almost certainly the concept of negative frequencies first comes up for electrical engineers when they encounter Fourier Transforms (FT). [To the extent that engineers may have first encountered the FT in an "applied" math course, where the paired variables are perhaps, x and y (assumed dimensionless), no conceptual difficulties were encountered. It's just math!]

Then, perhaps the engineer found to his/her surprise that the FT was really good for something. We could use it to provide alternative descriptions of a waveform (or "signal"), with the Fourier transform variables now becoming time and frequency. Here we do not propose to review the FT theory, and its many extremely useful applications. Instead we remind the reader that the continuous-time FT has the form of two HIGHLY SYMMETRIC integral equations [9] where both time and frequency ran from $-\infty$ to $+\infty$, the limits on the integrals. Accordingly, we have both negative and positive ranges of times, and negative and positive frequencies. How are we to deal with these negative ranges? How in fact are we to deal with zero values of these variables? Well, there is a pretty sound notion of an exact zero of frequency – direct current (DC) – a constant independent of time. But what about time? We are accustomed to assigning a zero of time arbitrarily, generally for convenience. Once we do assign a zero, negative time is rather obvious. But with the extreme symmetery of the time/frequency descriptions using the FT, it remains a puzzle (to me at least) why we don't have an exact zero, but only an arbitrary zero, of time.

But engineering students are likely to be further outraged by any notion of a physical meaning to a negative frequency. Indeed. Looking at a sinewave oscillator or function generator in a lab, you will find no negative frequency range. (And probably no zero either – ask for a generator putting out zero frequency, and they give you a power supply!).

A first approach to negative frequencies (see also Note 1 at the end) might be to consider a sinewave. It is true that:

$$\sin(-\omega t) = -\sin(\omega t) \qquad (1a)$$

which suggests that the negative frequency is just an inversion of the waveform. Fine until we remember that :

$$\cos(-\omega t) = \cos(\omega t) \qquad (1b)$$

An arbitrary sine wave (of unit amplitude for simplicity) is of course a combination of a sine and a cosine:

$$\sin(\omega t+\theta) = \cos(\theta)\sin(\omega t) + \sin(\theta)\cos(\omega t) \qquad (2a)$$

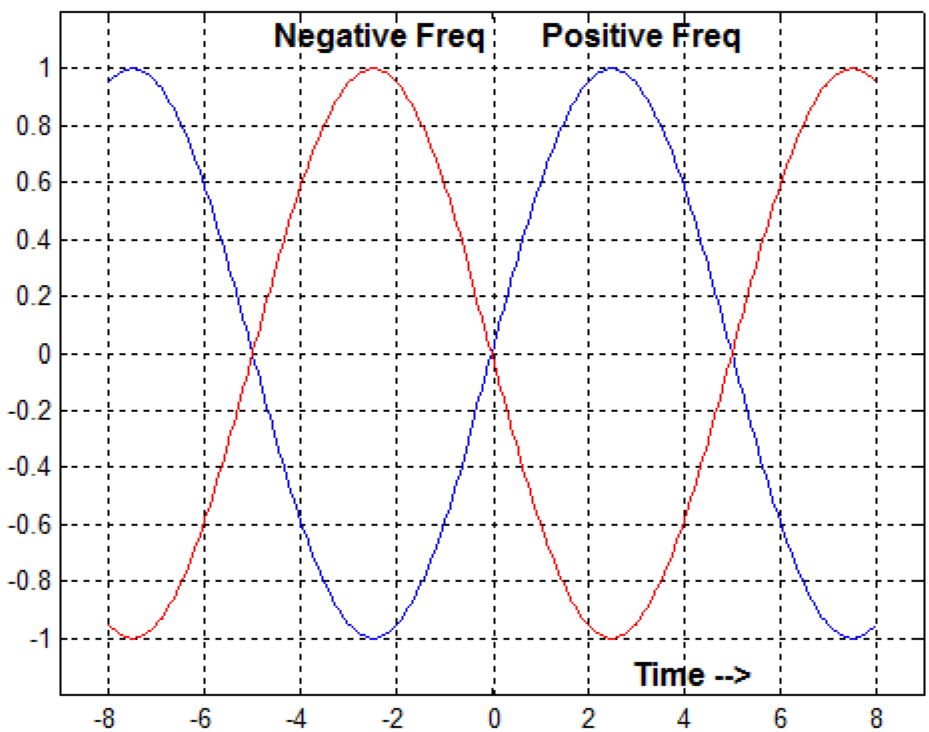

Fig. 1a Negative frequency version of sine is just an inversion [Equation (1a)]

where the multipliers in front of the time-varying sine and cosines depend on the fixed values of the sine and cosine of the phase angle θ. Accordingly for a negative frequency:

$$\sin(-\omega t+\theta) = -\cos(\theta)\sin(\omega t) + \sin(\theta)\cos(\omega t) \tag{2b}$$

From which we see that the difference between the positive and the negative frequency is simply that the weight on the sine component is reversed. It is most useful here to look at some graphs (see Matlab program **en206-1.m** at the end).

In Fig. 1a, we show a plot of $\sin(\omega t+\theta)$ and of $\sin(-\omega t+\theta)$ for the case of $\theta=0$ and $\omega=1$. In this case, as in equation (1a), we see only an inversion to be the result of choosing the negative frequency. Things will be different when θ is not 0. Fig. 1b shows the plot when $\theta=30°$. Clearly in this case, when $t=0$, equations (2a) and (2b) both yield $\sin(\theta)$ which is 1/2 in this case. Note that the negative frequency waveform is the positive frequency waveform mirrored about $t=0$.

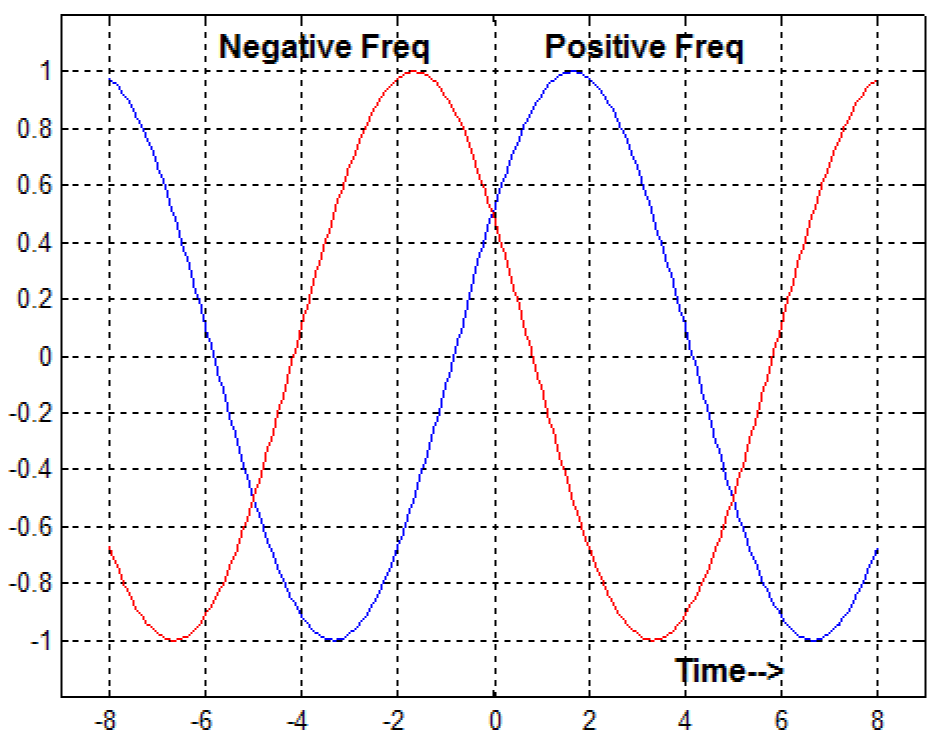

Fig. 1b Positive and Negative versions with θ=30°

Using the same program, we can set $\omega=0$ and leave $\theta=30°$. This "silly" illustration, Fig. 1c, shows that the result is $\sin(30°)=0.5$ everywhere. In fact, the blue line (positive frequency) was overplotted with the red line (negative frequencies) so the two are not only constant, but the same constant. While obvious, this shows the consequence of an important fact that we shall use below: Because there are only two ways to go from positive to negative – in a discrete jump, or through zero - we are interested in what happens when the frequency is zero. The waveform is flat.

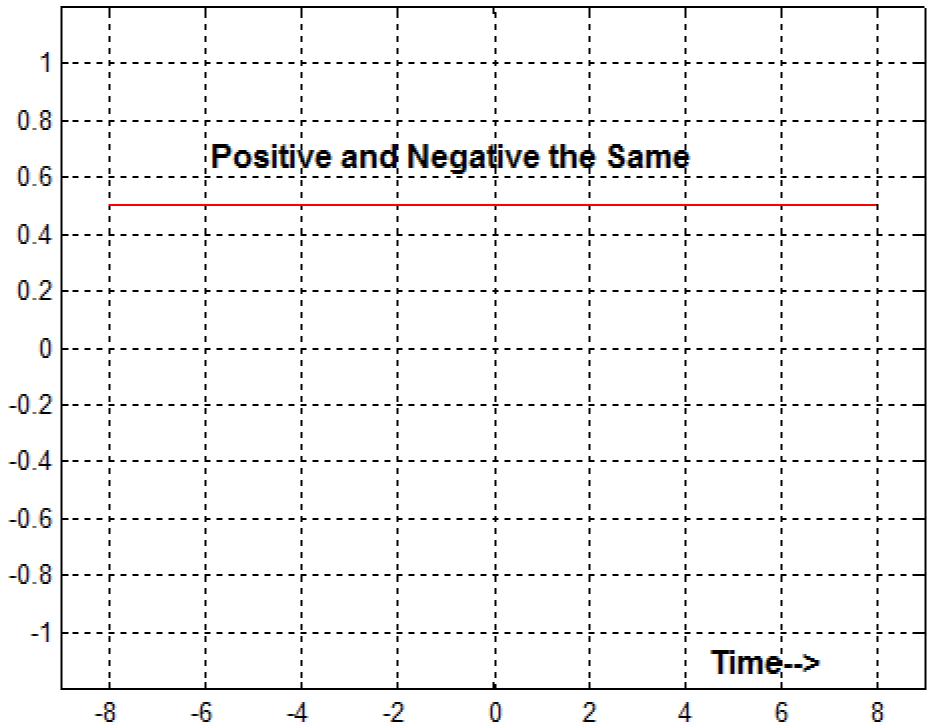

Fig. 1c With frequency $\omega=0$, and $\theta=30°$, both positive and negative are a constant 0.5

NEGATIVE FREQUENCIES AS A TIME REVERSAL

This idea we have considered in the past. It is another interpretation of negative frequencies. We begin with a notation that will be needed a bit later:

$$x(t) = \sin(A(t)) \tag{3}$$

where A(t) is a time-varying phase. (This is exactly the same A(t) which we have used in the past in our FM problems.) Now, one simple form of A(t) would be to have the phase be a linear function of time. That is:

$$A(t) = \omega t \tag{4}$$

where ω is a constant (a fixed frequency). Differentiating (4) we have of course just:

$$dA(t)/dt = \omega \tag{5}$$

That is, the frequency is the time rate of change of the phase angle. In a generalized sense, where A(t) is not just a simple linear function, we have the notion of an "instantaneous frequency"

$$\omega_{inst} = dA(t)/dt \tag{6}$$

which is so useful in FM problems (Fig. 2).

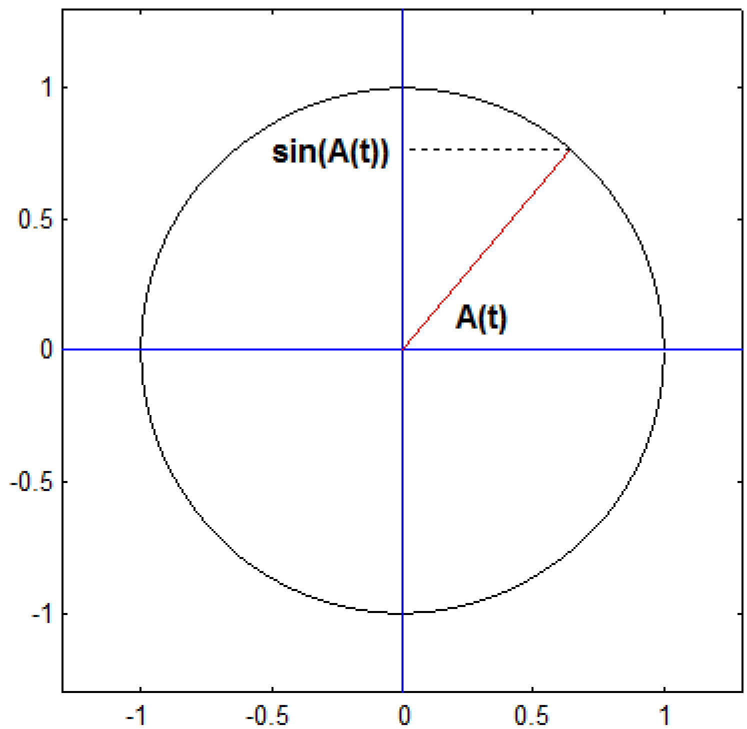

Fig. 2 The essential rotating phasor model of FM

From Fig. 2 we envision the "phasor" (red line) rotating at a constant rate for fixed ω, and at a variable rate for ω_{inst} (for FM). The sine of this angle A(t) is the actual signal.

For the moment, consider the case of a fixed frequency, A(t) = ωt, and θ = 0. Thus:

$$x_p(t) = \sin(\omega t) \qquad (7a)$$

and the corresponding negative frequency case would be:

$$x_n(t) = \sin(-\omega t) = \sin[\omega \bullet (-t)] \qquad (7b)$$

which corresponds to the case of which Fig. 1a is typical, and leads to an interpretation of the negative frequency as a reversal of the signal in time.

This in turn leads to an approach to a hardware realization. The question of the generality of this interpretation needs to be considered in the context of a discrete jump (past zero) to negative frequencies, and one that goes through zero.

DISCRETE JUMP TO NEGATIVE

Suppose we have a sinewave of frequency ω and it suddenly jumps to a frequency -ω. This would seem to be equivalent of jumping from one waveform to another, where the

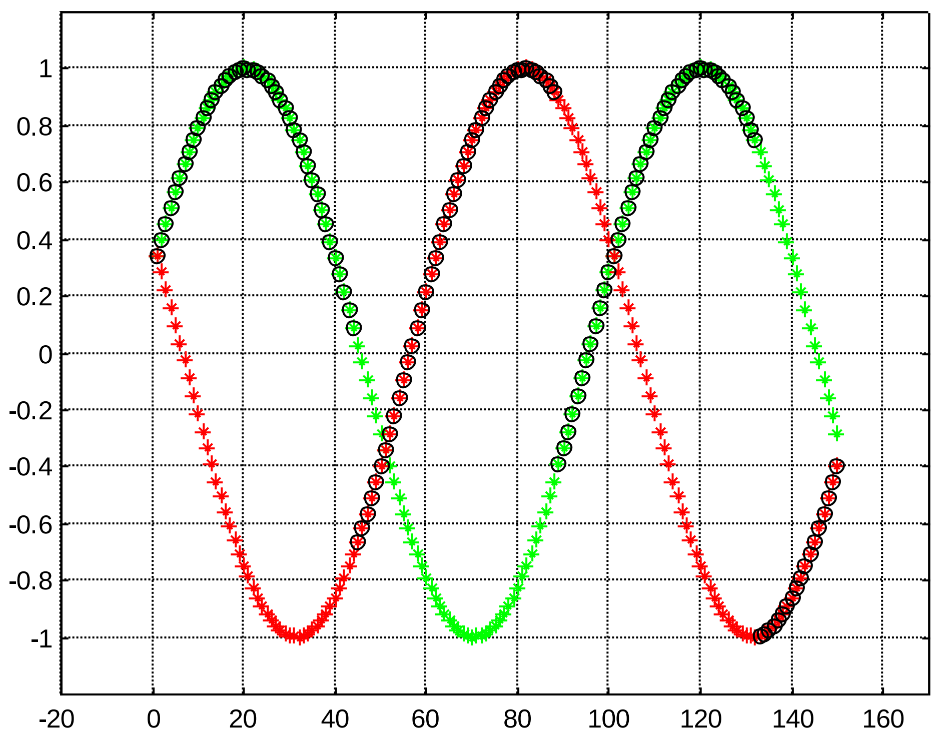

Fig 3 Discrete Jump between positive (green) and negative (red)

two waveforms differ in phase, as in Fig. 1b. In general, this means that there will be a discrete jump in the waveform. An example of what we are talking about is shown in Fig. 3, which was generated by Matlab program **en206p3**. In Fig. 3 we see two waveforms, a positive frequency (green) and a negative frequency (red). We show the waveforms not as a continuous waveform, but as stars. (This is because if we were to plot a continuous line, the points on either side of the jump would be connected by a slightly diagonal line, and this can be misleading.) The result is segments that are 44 samples long (shown by circling the stars), alternating between positive and negative. The question is – is this TZFM as we want to have it?

Well, It certainly is some sort of modulation. Perhaps you can regard it as the sum of two amplitude modulations, where the modulating waveforms are square waves (with levels 1 and 0) and the carriers are two sinewaves, of the same frequency, but of differing phase. That is, is there anything that would indicate that the two phases are related in a strong mathematical way by a positive/negative frequency relationship? It doesn't seem likely.

THROUGH ZERO TO NEGATIVE

To this point, we have been dealing with FM only to the extent of a single frequency (!), or to the extent of having a positive and a negative version of the same frequency. We now need to consider the real case, where the frequency is in general time varying and passes through (not jumping over) zero. Here we need to make use of the notion of the instantaneous frequency, equation (6). We now note expressly that ω_{inst} is a function of time, $\omega_{inst} \equiv \omega_{inst}(t)$. Integrating equation (6) we have the integral:

$$A(t) = \int_0^t \omega_{inst}(t)\, dt \qquad (8)$$

which we can approximate by a summation (for our numerical method of analysis):

$$A(M\Delta t) \approx \sum_{n=0}^{M} \omega_{inst}(n\Delta t)\, \Delta t \qquad (9)$$

Note that:

$$A(t) \neq \omega_{inst}(t)\, t \qquad (10)$$

Here M corresponds to the current time in a discrete form. This tells us to write our programs in a different way from what seemed (at first) intuitively right.

Looking again at Fig. 2, we need to think of the angle A(t) as being the result of integration. It moves around as a non-linear function of time (except for a fixed frequency) and is NOT just a calculated value at a point in time. Rather it is the result of integration (of the entire "History" getting there). This is a key point about doing FM problems right, as compared to just getting them "something like" reality. [Below we will show an example of doing it wrong - a real "mistake" that will illustrate how things can go wrong.]

So we have found that we should not think of a frequency as going through zero, because a frequency is a constant value. Instead, it is the instantaneous frequency that is time-varying, and the instantaneous phase A(t) is the integral of this instantaneous frequency. For the purposes of computing this using Matlab, we need to use the discrete summation, equation (9). The program we need is **en204p4.m** which produced Figures 4a – 4e, and it is printed in full at the end. A core portion of the code corresponding to the discrete summation is shown in the snippet at the top of page 12.

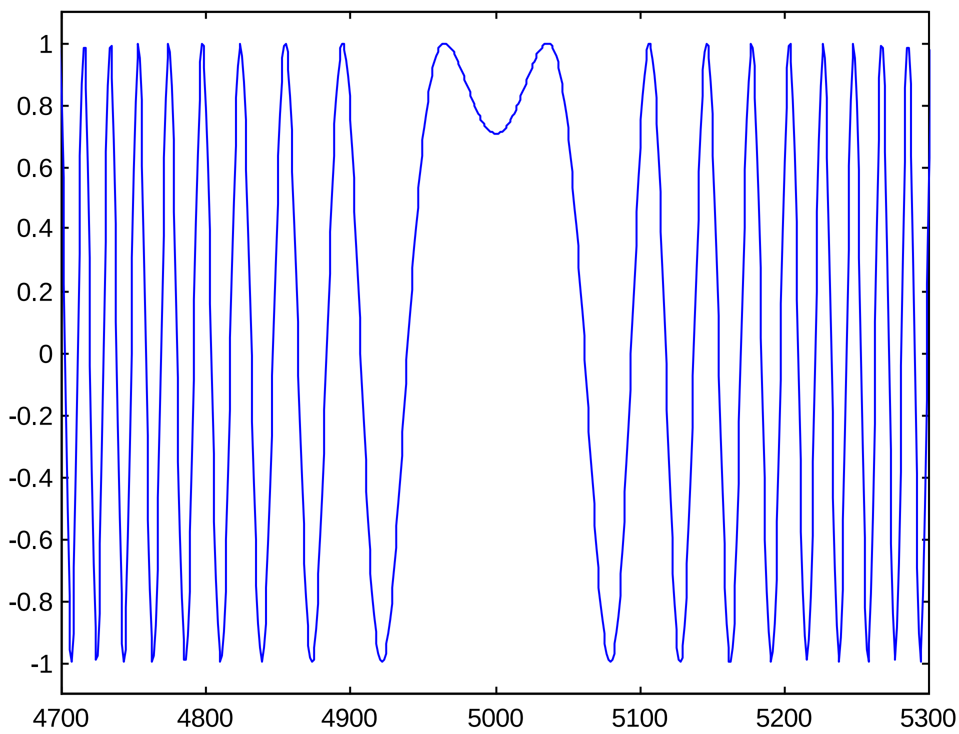

Fig. 4a Normal turn around ("time reversal") at zero instantaneous frequency (at sample 5000) is exactly what we hoped to find.

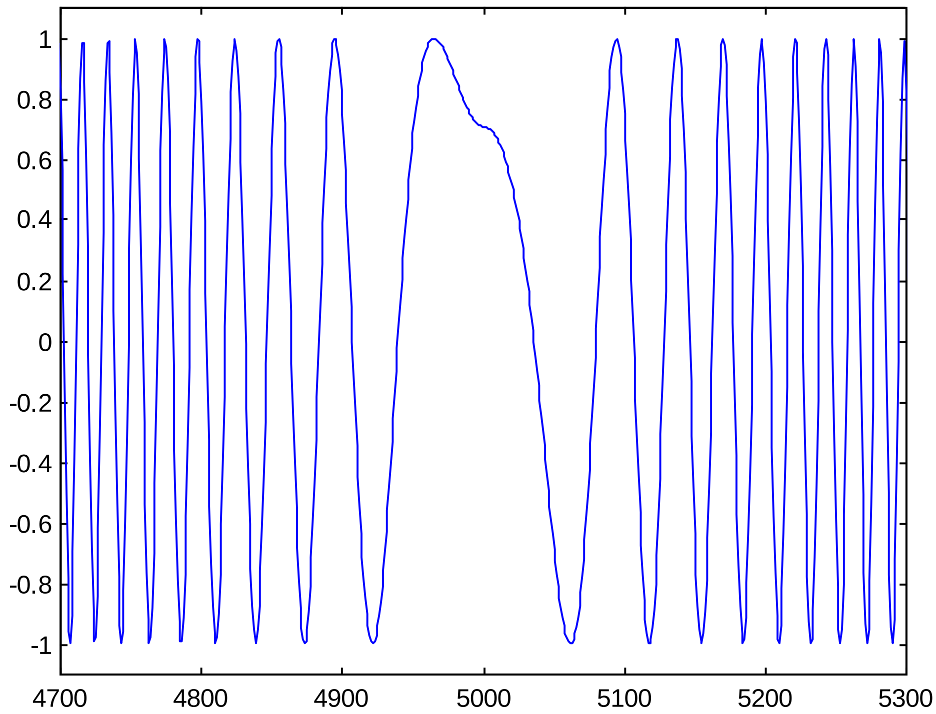

Fig. 4b Case where modulation does not go negative

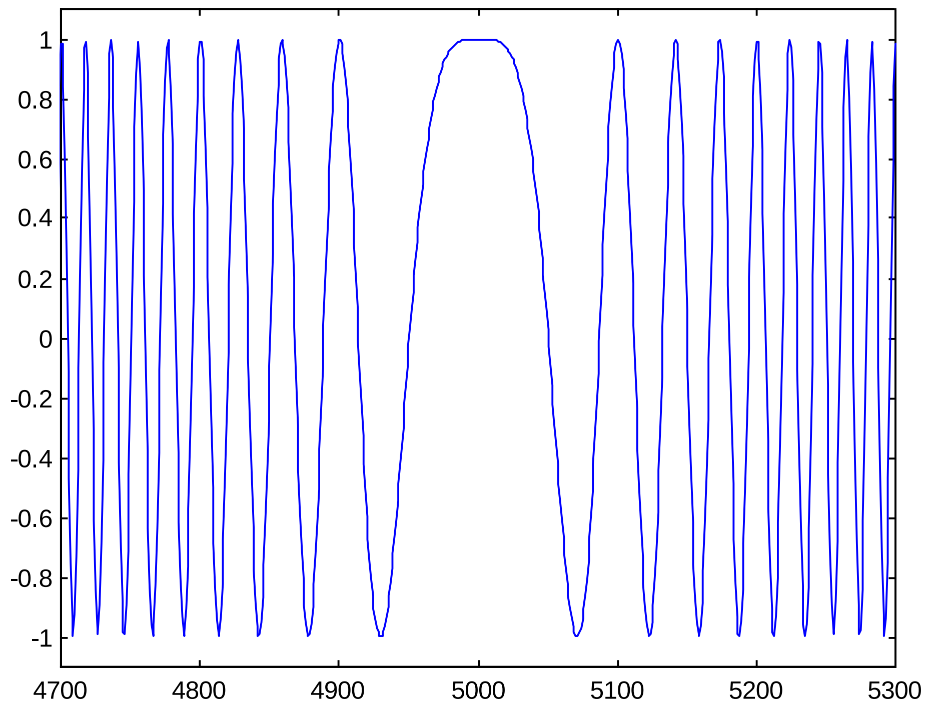

Fig. 4c Case here “turn-around” is exactly at top.

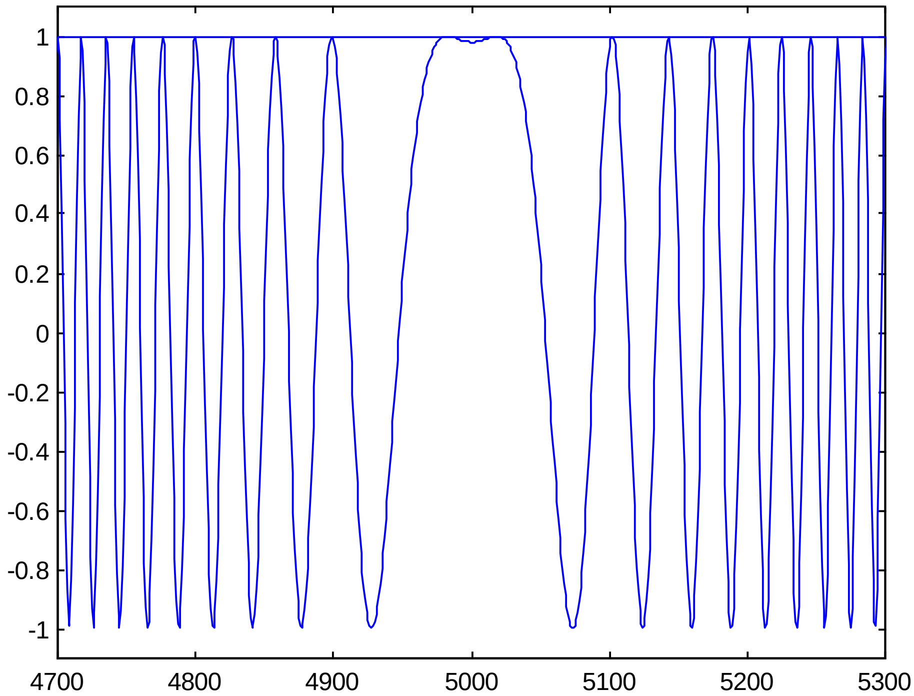

Fig. 4d Case where through-zero turn-around occurs slight after top turn around

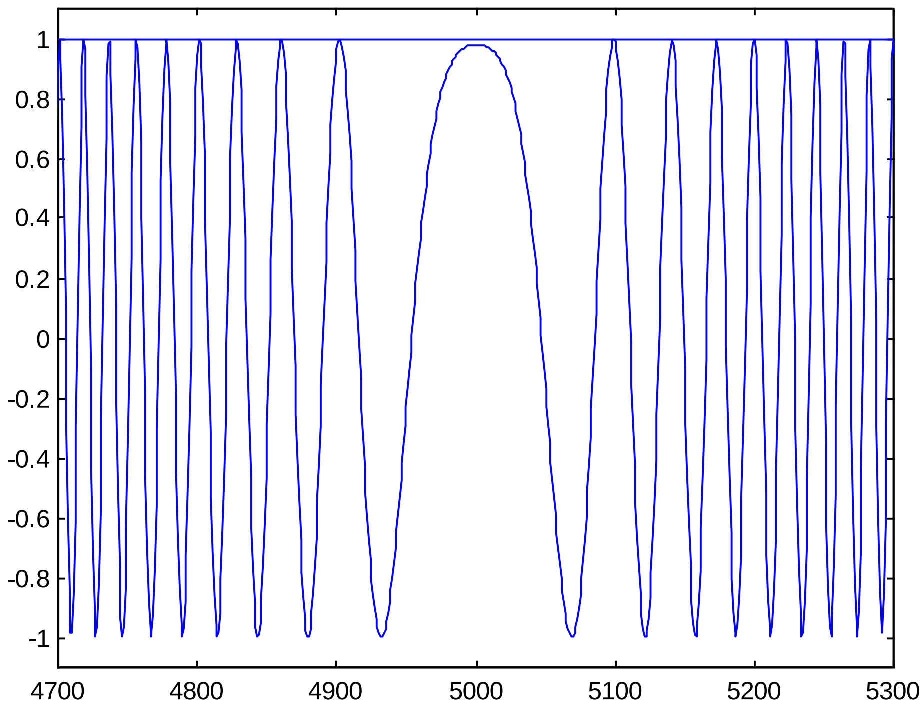

Fig. 4e Case where through-zero turn-around occurs slightly before top turn around

```
for n=1:10000;
   dAdt=2*pi - 2*pi*n/5000;
   A=A+dAdt;
   x(n)=sin(A);
end
```

Core Code Snippet

As the loop progresses with n, the instantaneous frequency (dAdt in the code, or in the snippet above) runs from 2π to -2π, and the increment is added to A. Then the corresponding sine of A is computed and placed in the time series. Figures 4a through 4e show the plots of the very middle portion of the outputs. (The longer signals were a remnant of listening tests which required more samples.)

Fig. 4a shows a typical through-zero turn-around in mid-cycle. This is a characteristic we have looked for with TZFM (including seeing such features fleetingly on an oscilloscope with analog realizations). There is necessarily a flat region (zero derivative) at the turn around region. In Fig. 1c we saw that a constant results from a zero in frequency. If however we have the instantaneous frequency going from positive to zero and then “bouncing off” and going again positive instead of negative, we get the result of Fig. 4b. The waveform slows and then proceeds in the direction it was previously going, unlike the turn-around in Fig. 4a [see also Note 2 at end].

One puzzling case is illustrated in Figures 4c, 4d, and 4e. In the TZVCO case, or just by considering Fig. 4a, it is clear that there are two ways the waveform can reverse direction**:** by the waveform reaching its amplitude limits, or by the modulating signal going through zero. What if both happen at once? Well, one way to look at it is to not think of these two conditions as telling the generator to reverse direction, but rather that there are <u>two instructions to do the same thing</u> – to move away from the peak. Fig. 4c shows such a case where the zero-crossing of the instantaneous frequency is at the very top and results in a broad, relatively flat region about the positive limit. More insight into what happens can be seen From Fig. 4d where the zero-crossing reversal occurs slightly after the natural bounce off the rail, and this is followed quickly by a second bounce. Fig. 4e shows the case where the zero-crossing reversal occurs very close to the top rail, but the waveform is turned back without touching. Thinking of Fig. 4c as the case between Fig. 4d and 4e should lead to the correct perspective.

HETERODYNING ALSO SHOWS TURN-AROUNDS

Here we will shows a result that should be convincing with regard to the notion of time-reversal being the result of through-zero modulation. As we said, use of a frequency shifter to achieve TZFM was a straightforward application of the shifting device [5]. On the other hand, it is less clear intuitively that the TZVCO does the right thing. So, one test (of the TZVCO) is to see what the frequency shifter (a result we believe) does in the time domain. Do we see the turn-arounds?

The frequency shifter (also called a "Single-Sideband Modulator") is classic, and had appeared in ***Electronotes*** many times. Fig 5 below is one such instance from a recent App Note [10]. This is one form of a more general "heterodyning" method. The product of two sine waves of two different frequencies is two different sinewaves superimposed, one at the sum frequency, and one at the difference frequency. This is just a trig identity of course (shown in the figure). We are interested here in looking at the difference frequency as the difference can be negative as well as positive. For example, the "beat oscillator" ("quadrature" components xs and xc) might be a fixed 1000 Hz while the variable oscillator ("quadratuer" components ys and yc) might vary from 1200 to 950, so the difference would vary from +200 to -50 Hz.

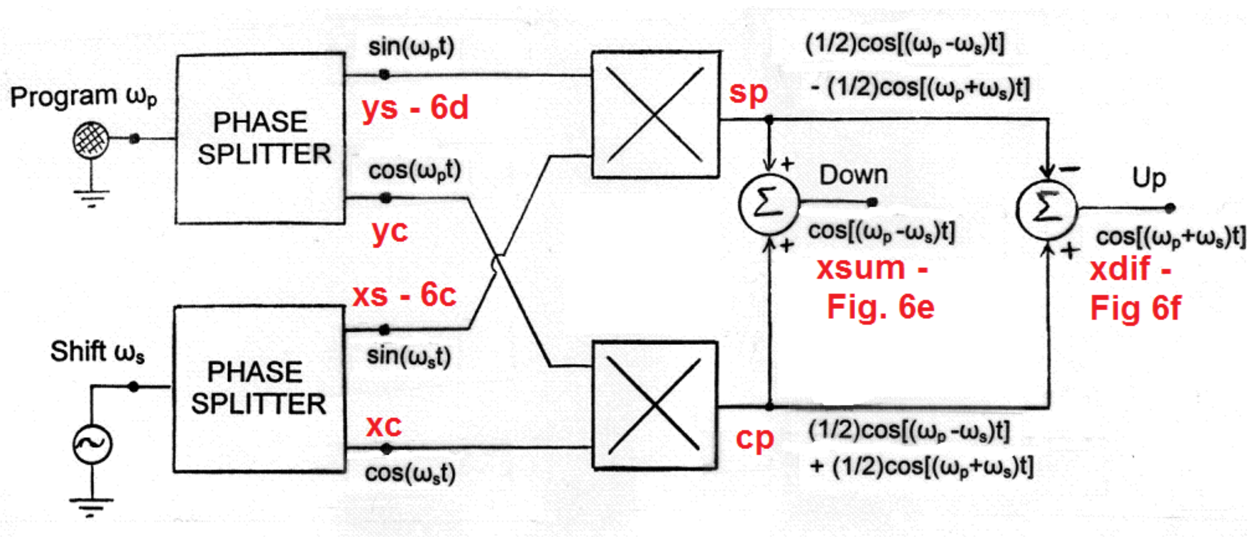

Fig. 5 A classic frequency shifter as used in electronic music

For our purposes, we don't need the phase splitters, as we will just generate sine and cosine sequences in Matlab. Then it is a matter of multiplying the sine together, and the cosines together, and adding the two products. Very simple – shown in the program **en206p6.m** at the end.

A number of years ago, thinking about doing this experiment, I typed out some Matlab code and ran it. I was looking for the classic time-domain turn around, and was delighted that it immediately appeared (as in Fig. 6a). Good enough? Well, perhaps, but it was wrong, as I had made (not for the first time) the classic mistake described above.

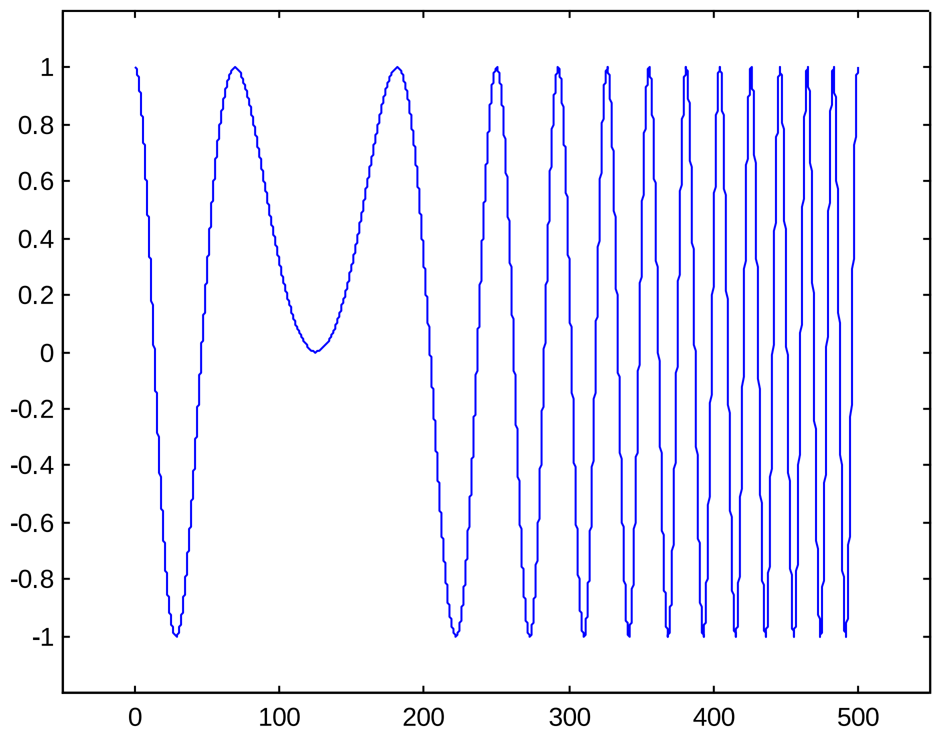

Fig. 6a It was delightful to see the time-domain turn-around, but the turn-around should have been at sample 250 !

The result in Fig. 6a is actually "correct" in demonstrating that the through-zero is a time-domain turn-around. Seeing that and not having more time, I left it at that a few years ago. Recovering the script file for this current investigation, I did see that something was wrong. The turn-around should have been at sample 250, and instead, it is near sample 120. I had arranged for the frequency to be 0.1, the same as the beat frequency at 250, and indeed it was (see Fig. 6b).

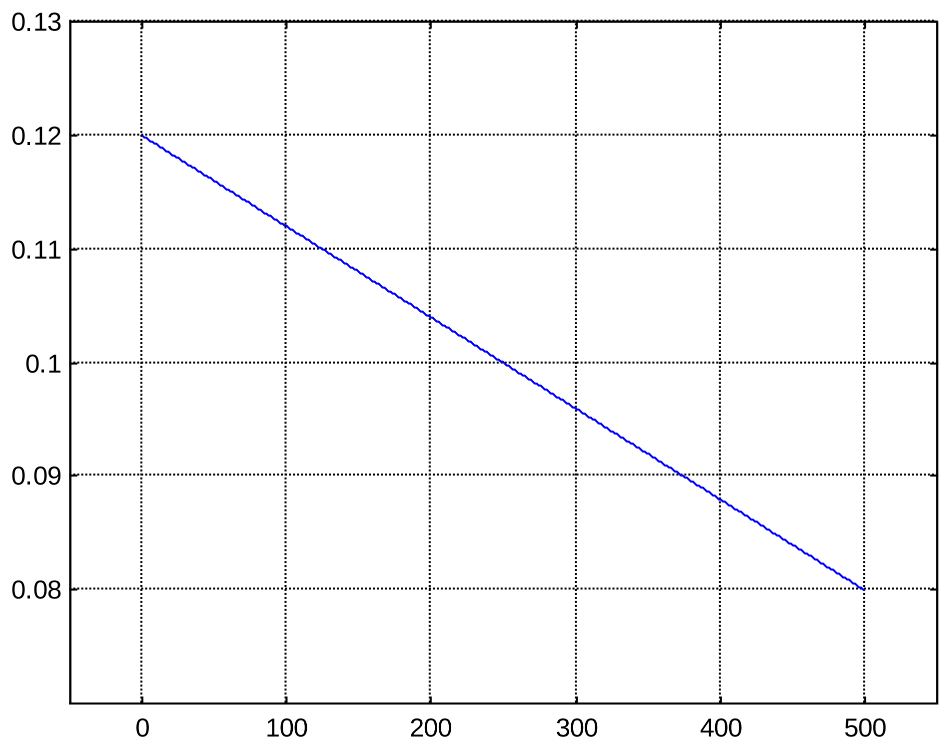

Fig. 6b The parameter “f” of the Matlab program – looked okay

By plotting the x and the y sinusoidal waveforms, it was clear that the y waveforms were about the same apparent cycle length as the x somewhere about sample 120. That is, by eye, the x sinusoidal waveform was a constant frequency, while the y sinusoidal started as a higher frequency, and ended at a lower frequency. The place where they were apparently the same was not near 250, but more like 120. Modify and run **en206p6.m** as described below to see this if you so desire. Well, that explained why the turn-around was there near sample 120.

The mistake was of course the failure to recognize that the parameter f, here carefully set as in Fig. 6b, is not the frequency, but the instantaneous frequency. As we did correctly in our Fig. 4 examples, this needs to be integrated to get the phase A(t). The program **en206p6.m** actually represents two programs – doing it right, and doing it wrong, according to which lines are commented in or out, as indicated. Fig. 6c shows

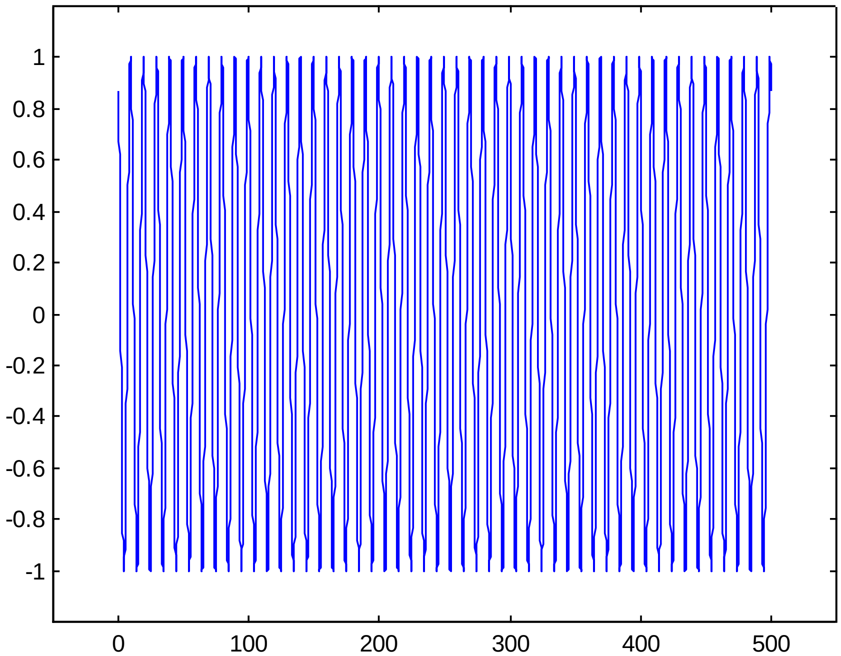

Fig. 6c The beat oscillator (xs of program)

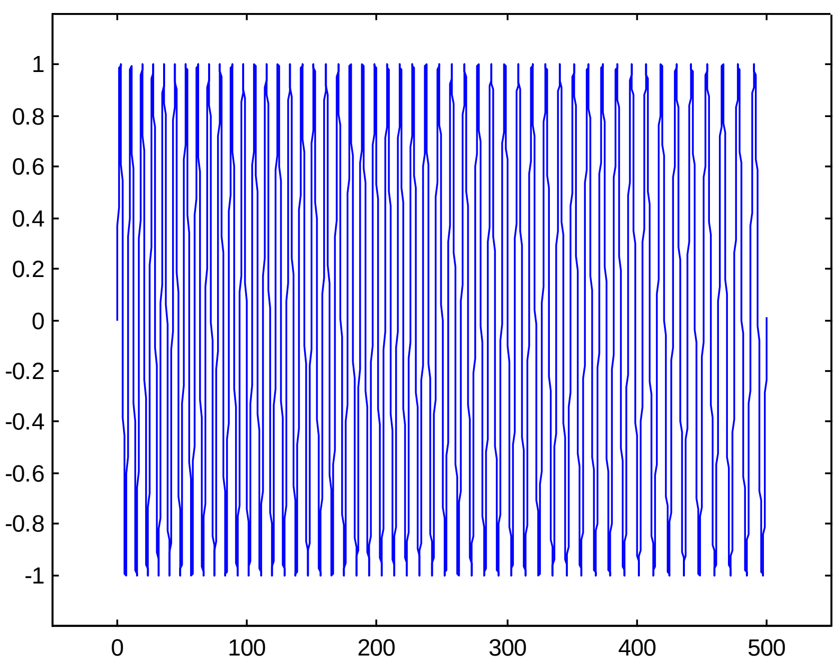

Fig 6d The shifting oscillator (ys of program)

the waveform of the "beat oscillator" (xs, the sine component in the program – the cosine component xc looking very similar of course). It is a constant frequency sine. The sine component of the moving oscillator is shown in Fig. 6d (ys, the cosine component yc looking very similar). Note that the periods in Fig. 6d are about the same (visibly) as those of Fig. 6c near the middle, as they should. The real proof is Fig. 6e, where we show the difference frequency. Indeed, we have the expected time-domain turn around, this time in the middle. as it should be.

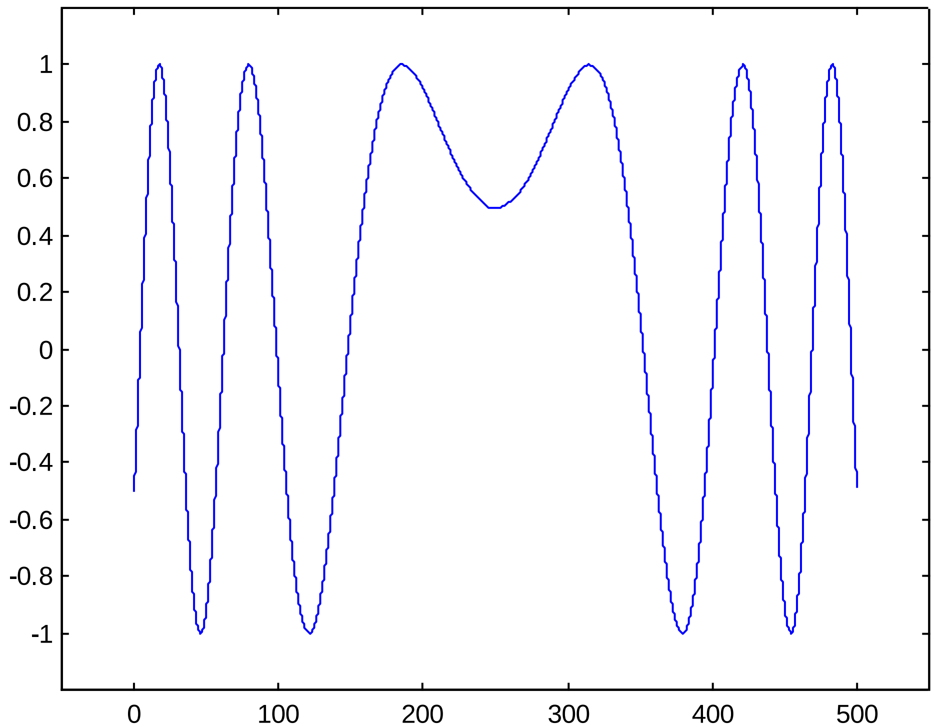

Fig. 6e The downshift done correctly

For completeness, we also show the upshifted version in Fig. 6f. Clearly it is a much higher frequency. The point of showing this is to indicate that in such a case where we may desire only the downshift (difference frequency) the two, which are in the outputs of both multipliers, may be separated by a low-pass filter. This is what we saw in the digital oscillator case. It was also the strategy in Jan Hall's circuit [6].

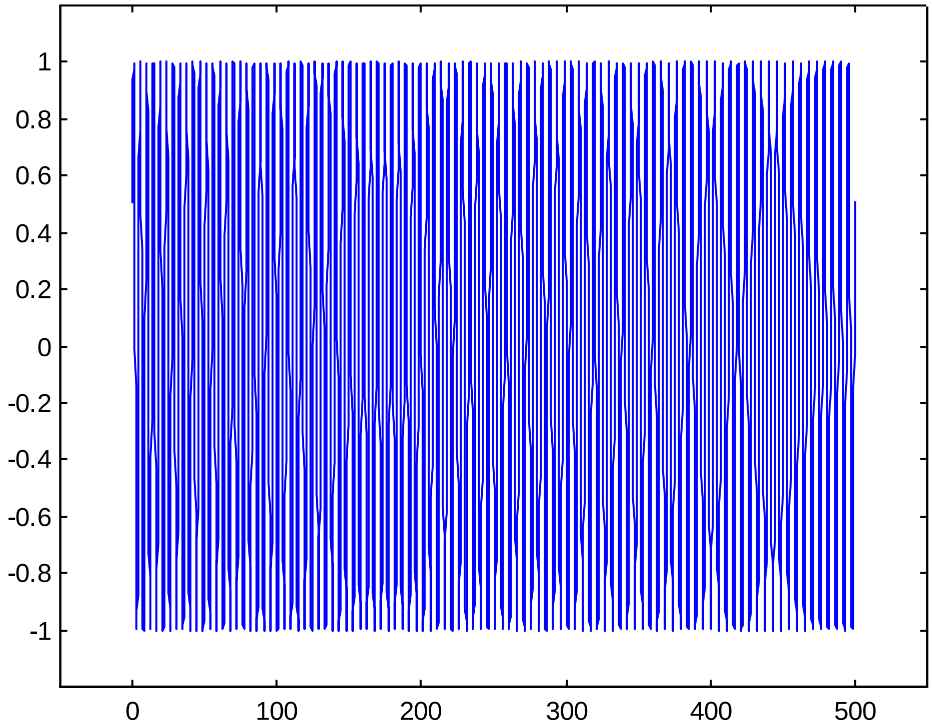

Fig. 6f The upshifted signal

CONCLUSIONS

We conclude that the notion of a time-reversal as representing a through-zero FM project is valid in terms of all the usual realizations, including a direct digital (like these Matlab calculations), a frequency shifter, a through-zero VCO, and a digital oscillator. There is a warning that some care must be taken, with respect to considering an instantaneous frequency to be the time derivative of phase, or something "like" what we are expecting, rather than the true result, may be found.

NOTES:

NOTE 1:

While we have been talking about a negative frequency, or perhaps a negative instantaneous frequency, the notion of negative frequencies as components of an FM spectrum is somewhat different and possibly familiar to our readers. Because the amplitudes of FM sidebands are controlled by Bessel Functions, which decay asymptotically away from the carrier, in theory, we always have negative frequencies in the spectrum. With music sound synthesis (as opposed to radio), this is a fundamental issue, because the carrier and the program are pretty much both in the audio range. We know about negative frequency spectral components folding back and being heard as ordinary audio. The connection with through-zero instantaneous frequency is as follows. Significant FM spectral components are generally found only in a range where the region is "swept" by the instantaneous frequency. Thus, negative frequency spectral components are most likely audible only with TZFM, where the negative range is swept.

NOTE 2:

While we have been concentrating on a single "cycle" of modulation (the case where the instantaneous frequency goes through zero once) , in the generation of musical sounds, the modulation goes through zero many times during the sounding of a single musical note. Further, the depth of this modulation varies during the note. For TZFM (indeed for all FM) to be the most useful, we need this dynamic-depth through-zero FM. The note may begin with shallow depth but rapidly reach deep depth (diring "attack"), and then more gradually this depth decreases (during "decay"). This means that the note opens fairly rapidly into a very rich spectrum, and then spectral components simplify. This is the way many acoustic instruments behave.

As the depth increases, we hear the spectral enrichment, and this continues as the instantaneous frequency, on consecutive modulation cycles, goes lower and lower, to zero, and eventually through zero. Nothing "special" happens audibly as the depth goes through zero. Going 1% below zero is much the same as going from 1% above to zero, as for the perceived nature of the change. If you hear anything quite different suddenly occurring, something is wrong. Fig. 4b (a failure to reverse), for example, would sound different as the zero transition took place. Some folks think something is wrong with a TZVCO if they hear nothing special happening as the depth crosses the zero frequency. It's just a continuation of the above zero enrichment.

Overall, using TZFM you can achieve a much much richer variety of sounds, as compared to just going to zero.

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[8] Hutchins, B., “The ENS-76 Home-Built Synthesizer System – Part 7, VCO Options,” Electronotes, Vol. 9, No. 75 (March 1977) (see Option 4); Hutchins, B., “A VCO with Through-Zero Frequency Modulation Capability,” Electronotes, Vol. 13, No. 129, (Sept. 1981) pp 3-12;

[9] See Hutchins, B., “Transform Methods in Musical Engineering,” Electronotes Supplement S-015, July 1977. OR see any of numerous excellent books on the FT, such as the book by Bracewell.

[10] Hutchins, B., “Phase Errors, Amplitude Errors, and Unwanted Sidebands in Frequency Shifters,” Electronotes Application Note 352, May 2002

PROGRAMS

en206p1.m

```
% en206p1
% generates Fig. 1, a,b, and c
% Simple interpretation of negative frequencies

t=-8:.05:8;
figure(1)
w=.1
theta=0
xp=sin(2*pi*w*t+theta);
xn=sin(2*pi*(-w)*t+theta);
plot(t,xp,'b');
hold on
plot(t,xn,'r');
hold off
grid
axis([-9 9 -1.2 1.2]);

t=-8:.05:8;
figure(2)
w=.1
theta=pi/6
xp=sin(2*pi*w*t+theta);
xn=sin(2*pi*(-w)*t+theta);
plot(t,xp,'b');
hold on
plot(t,xn,'r');
hold off
grid
axis([-9 9 -1.2 1.2]);

t=-8:.05:8;
figure(3)
w=0
theta=pi/6
xp=sin(2*pi*w*t+theta);
xn=sin(2*pi*(-w)*t+theta);
plot(t,xp,'b');
hold on
plot(t,xn,'r');
hold off
grid
axis([-9 9 -1.2 1.2]);
```

en206p3.m

```
% en206p3
% Produces Fig. 3
% Discrete jumps betweeen positive and negative.

t =0:9999;
theta=pi/9
xp=sin(2*pi*t/100+theta);
xn=sin(-2*pi*t/100+theta);
s=[ones(1,44), zeros(1,44)];
   s=[s s s   ];
   s=[s s s s s];
   s=[s s s s s s s s s s s];
   s=s(1:10000);
s1=1-s;
s1(1:50);
xps=xp.*s;
xns1=xn.*s1;
xsum=xps+xns1;

figure(1)
plot(xp(1:150),'g*')
hold on
plot(xn(1:150),'r*')
hold off
grid
% This will be Fig. 3 of text
figure(2)
plot(xp(1:150),'g*')
hold on
plot(xn(1:150),'r*')
plot(xsum(1:150),'ko')
hold off
axis([-20 170 -1.2 1.2])
grid
```

en206p4.m

```
% en206p4
% This program produces Fig. 4,a,b,c,d,e
% Direct computation of phase A(t)
% Initial phases A chosen for clear illustration of point to be made
% Longer length of signals (relative to amount plotted) was for
%     listening purposes using sound (not in program here)
A=-pi/4
x=[]
for n=1:10000;
   dAdt=2*pi - 2*pi*n/5000;
   A=A+dAdt;
   x(n)=sin(A);
end
figure(1)% 4a of text
```

```
plot(x)
axis([4700 5300 -1.1 1.1])


A=-pi/4
x=[]
for n=1:10000;
   dAdt=abs(2*pi -  2*pi*n/5000);
   A=A+dAdt;
   x(n)=sin(A);
end
figure(2) % 4b of text
plot(x)
axis([4700 5300 -1.1 1.1])


A=3*pi/2
x=[]
for n=1:10000;
   dAdt=2*pi - 2*pi*n/5000;
   A=A+dAdt;
   x(n)=sin(A);
end
figure(3) %4c of text
plot(x)
axis([4700 5300 -1.1 1.1])

A=3*pi/2+.2
x=[]
for n=1:10000;
   dAdt=2*pi - 2*pi*n/5000;
   A=A+dAdt;
   x(n)=sin(A);
end
figure(4)  % 4d of text
plot(x)
hold on
plot([4700 5300],[1 1])
hold off
axis([4700 5300 -1.1 1.1])

A=3*pi/2-.2
x=[]
for n=1:10000;
   dAdt=2*pi - 2*pi*n/5000;
   A=A+dAdt;
   x(n)=sin(A);
end
figure(5) % 4e of text
plot(x)
hold on
plot([4700 5300],[1 1])
hold off
axis([4700 5300 -1.1 1.1])
```

en206p6.m

```
% en206p6
% Produces Fig 6      of text
t=0:.1:500;
length(t)
theta=2*pi/3;
%theta=0
xs=sin(2*pi*(1/10)*t+theta);
xc=cos(2*pi*(1/10)*t+theta);
figure(1)
plot(t,xs)
axis([-50 550 -1.2 1.2])
figure(2)
plot(t,xc)
axis([-50 550 -1.2 1.2])

f=0.12:-0.04/5000:0.08;
length(f)
figure(7)
plot(t,f)
axis([-50 550 0.07 0.13])
grid

%ys=sin(2*pi*(f.*t));
%yc=cos(2*pi*(f.*t));
% to do it WRONG, remove the comments from the two lines above, and
% comment out the xis lines below.  And set theta=0 at top.
A=0;
for k=1:5001
   ys(k)=sin(2*pi*A);
   yc(k)=cos(2*pi*A);
   A=A+f(k)*0.1;
end
figure(3)
plot(t,ys)
axis([-50 550 -1.2 1.2])
figure(4)
plot(t,yc)
axis([-50 550 -1.2 1.2])
sp=xs.*ys;
cp=xc.*yc;
xsum=sp+cp;
xdif=sp-cp;
figure(5)
plot(t,xsum)
axis([-50 550 -1.2 1.2])
figure(6)
plot(t,xdif)
axis([-50 550 -1.2 1.2])
```