# ELECTRONOTES 200

NEWSLETTER OF THE

MUSICAL ENGINEERING GROUP

1016 Hanshaw Rd., Ithaca, NY 14850

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## GROUP ANNOUNCEMENTS

Contents of EN#200



Here we have issue No. 200. Enough said about that. The main contribution here is that we get around to the "Sampling Element" of our "Basic Elements of DSP" series, which we have managed to fit into a single newsletter. A good number of separate sampling examples were separated out and should eventually appear as part of a future newsletter, or as an application notes.

## Further Adventures With Analog Synthesizer Newsgroups

Some years back, as reported in an editorial "It's Better than TV" (EN#188, Feb. 1997) we had a run-in with some people on the internet who were posting schematics from our newsletter without permission and often without attribution. Issues raised went from copyright infringement to plagiarism to simple bad manners. All and all, the exchange was quite silly. Little is resolved in these newsgroup exchanges because any thoughtful contribution seems to spawn a half dozen weaker minded scrawls. (What was that cartoon? - On the Internet, no one can tell that you're a dog.) This was part of the criticism of the editorial, which had a serious side as well as a good-humored side.

Most of this squabble was eventually posted (even private emails!) on the "Analogue Heaven" or the "Synth-DIY" sites. I know that a number of the major players disappeared from the scene, and I lost track of the Synth-DIY site. I just couldn't find any way of seeing the messages without subscribing, and I have more than enough email without inviting more in. In early November of 2001 I stumbled on the excellent archives for Synth-DIY (which does not seem to be working for the last couple of weeks as of this writing in early Dec. 2001): at http://www.buchi.de/SDIY/. I spent an entertaining hour or so going over a few years of messages relating to Electonotes. The incentive to surf was, as is usual for me, a pile of ungraded exams that needed ignoring. (Hey Look! The lawn needs mowing!) Here are a few of the things I would have been tempted (only tempted) to respond to if I had seen them when they first appeared.

(1) First, there was some discussion of having Electronotes available on a CD. Either we or someone else would make it available on some terms and conditions that might or might not be agreeable or even practical. There is not the slightest doubt that this would be the cheapest and most convenient form to mail it about the US and the world. But this is the only thing about the idea that is clear in even just its practicality.

Most people of my acquaintance (and age and eyesight!) do not like to, or cannot read documents on a computer screen, nor does everyone have a computer with a CD ROM drive. For myself, I can write on a screen but I can't proofread (or necessarily read something someone else wrote) on a screen. I need a hard copy in good light, and as importantly, I need to be able to scribble on the hard copy if I am to make actual use of it. I absolutely hate to scroll through a long document on a screen looking for something. Generally a book can be opened to a relevant section in seconds. Browsing through a book involves moving the eyes naturally <u>left and right</u>. Browsing though a document on a screen takes forever and gives one a headache. The "paper" moves and the eyes (and or head) move <u>up and down</u> trying to follow. If the scroll speed is fast enough to not take all day, it is so fast that an extra millisecond hesitation on the mouse key leads one to wonder if a whole paragraph (or even several pages) have disappeared unnoticed into the floor or ceiling!

In consequence, I think a CD would only be of use in certain cases. It would be useful to people who do have the talents and perceptual facilities to view documents on a screen. Presumably some such people exist. It would also be useful to someone who was trying to save shipping charges who has an economical means of converting the whole thing to paper on the receiving end. (Because this would be about 8 hours of laser printing for one sided copies, this seems an unlikely choice.) The only remaining scenario would seem to be people who would only want to print out a few pages or who just wanted the collection for possible reference. That is, people who do not want to read it in the sense of actual use or study.

Another interesting question is what would be on the disk. Presumably - everything. But a few months after the disk (or run of disks) was made there would be new material to add. How does this join its predecessors? Do you order a new disk. Does it come over the internet?

Would the CD be secure, or would it soon be pirated to the point where few people would care to pay for it rather than just grab a free copy that is going by? Sounds like the MP3 scenario What would happen to my attempts to sell paper copies even if rampant pirating did not occur? I don't know [Is this CD a legal or an illegal copy? Right now, any CD of our material would be illegal. That's simple ]

Now, along with the suggestions that a CD be made available, some people commented on what they saw as the obvious advantage to me. First, they said I would not have to stand in front of the copy machine whenever a new order came in. Wow! Do the math! I estimate that we have nearly 6000 pages in the "Everything" package. If I could slap an original on the glass every 10 seconds, it would take 17 hours to make a full set. And we have not done our own copying for about 15 years, so we would have to pay someone to do the copying. And the originals are not in great order or always in good repair. There are four file drawers of originals. Even an agnostic would say a prayer while approaching this cabinet (please, please be there!)

Of course we do not make copies one at a time. Most of the materials we have on the shelves or boxed away. This overprinting of older materials is, incidentally, why we can still offer the everything package for less than 5 cents/page. (These stocks, along with the time we took to produce the intellectual material, constitute our investment in Electonotes. And Electronotes still owes us! We have to get more back.) Making up a full order actually takes only 20 to 40 minutes. This does not suggest that we do not reprint. Just about every full order turns up an item or two which we have run out of. This leads to the problems of finding (and often repairing) usable originals. But - I only have to locate and repair selected items - not the whole 6000 pages - at any one time. Then there is the new investment in printing a couple of years worth. Yes - a CD would be easier - if it did not kill the paper business, or people did not want paper copies. Posters suggest that the CD business would be lucrative. What do you think?

Here is how we can think this through. Let me make the following offer to any responsible person: I will enter into a contract with you, and offer you the exclusive rights to produce and sell CD's containing the available Electronotes materials. You can sell as many as you want at whatever price you want. I will provide you with a full set of copies for your originals.

You will agree to the following: You will pay me $X on the signing of the contact and $X on the anniversary of the signing for a minimum of five years. You agree that I can continue to sell paper copies, and that Electronotes and I hold all copyrights. You agree that all your advertising and promotion will make it clear that with regard to the CD's, the customer is dealing exclusively with you, not with Electronotes, and Electronotes has no obligation to them. In short, for a yearly fee, you get the "lucrative" CD business! You cover your expenses (scanning, CD production, postage, bad checks, etc.) and make your yearly license payment to me, and you keep all the rest as profit (or perhaps you are doing this for free - it's up to you).

So - - - What is X? Clearly, to me X is the amount of profit I would expect to loose for not making paper sales. Fair enough? I have a reasonable way of estimating this. For you, X is an amount you expect to be able to pay me and still have a successful business. This is a matter of estimating your sales, your expenses, and setting your prices. I do not have much of an idea about this - hopefully you do. Do you suppose that X should be zero? Well, keep in mind that I do not watch TV, but I presume there are still people on late at night telling you you can make a fortune with no money invested. If you believe them, go with them - please not with me. (Some investment is necessary - for example, as mentioned, I have to pay up front for the paper copies which sit on my shelves.) 5X is your "at risk" investment, your incentive, which will also assure that you will take measures to prevent pirating of the CD, because you still have to pay me for five years even if something goes wrong. It is just a matter of trusting the ciber world.

The world is full of what seem to be good ideas. [Most universities have a way for students to suggest ways of improving lectures. The single most intelligent suggestion I have seen along these lines was that we should have a complementary donut and coffee tray at the door (a 9AM class!). Alas - no it did not happen. ] Let's have a CD sounds like another good idea. But in suggesting that someone else do it, I am asking other people to think it all the way through. And clearly considered, I am asking them to take the risk (I have not suggested royalties), and indeed to some extent, to insure me against their getting tired, or not protecting the product.

But this is a serious offer. If you are interested, please submit a business plan and your notion of what X might be. If you are inclined to pass this offer along to others, do not paraphrase or abbreviate it, as there are at least three important subtle clues in the text, put there for the serious reader.

(2) More distressing, there seem to be a few comments (unfavorable) about the way Electronotes does business, and this is extremely annoying to still have this going on. This will stop, or we will take steps to put a stop to it. One basic question is: is there anyone who has sent us money who has not received his or her material. Without question, the answer to this is YES! At least once a year, we pack up a newly arrived order, and send it promptly, correctly addressed, and it comes back! You figure it out. It happens to all mail order businesses. Do people write the wrong address on the form? Do they move suddenly? There are almost certainly other packages that do not get delivered correctly which do not come back to us, and we do make shipping errors (see "Troubleshooting" below). The USPS is indeed excellent, but not perfect. We had a couple of problems recently. A former collaborator and dear friend

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# INTRODUCTION TO THE SAMPLING ELEMENT

Probably every field of scientific or engineering study has one fundamental principle which can be pointed out as being essential for a useful understanding of all its subdivisions. Cosmology has general relativity, biology has evolution, geology has plate tectonics, and so on and on. DSP has Sampling. The "Basic Element of DSP" in this issue is the Sampling Element. Logically it might have come first, but for a number of reasons (e.g., the desired continuity with the analog filtering material) we did the Digital Filtering Element first, in three installments.

The material here, presented in a single installment, was first written about two years ago, and has undergone some revision. Since that time, a number of additional problems and/or examples have been developed which might have been included - but it is long enough as it is. These additional materials will likely appear in this newsletter at a later date.

As with the digital filtering material, it was our intention to briefly revisit the conventional introductory material, and then to take the subject further into the ominous "intermediate" level. We anticipate that readers have some basic ideas about sampling, and we want to "look into the corners" a bit more

# Basic Elements of Digital Signal Processing

# Sampling Element

-by Bernie Hutchins

## 1. INTRODUCTION TO SAMPLING

The sampling of a signal seems to be a relatively simple idea. We often sample in real life. For example, we may check the thermometer periodically, every hour say, to be aware of the temperature outside. If it is 20 degrees at 8:00 and 30 degrees at 9:00, we might guess that it was about 25 degrees at 8:30 (an assumption on the smoothness and slowness of transitions). In practical terms, these discrete samples and additional information we infer from them are often sufficient for our purposes. We would think it silly to sit by the window staring at the thermometer continuously. Similarly, we are generally sufficiently aware of the time of day through irregular glances at the clock. That is, sampling may still works even when the samples are irregularly spaced. (In this second example, how would you even manage to look at the clock at regular intervals? Watch another clock?)

Against this everyday background experience, the actual mathematics of sampling may be cumbersome. We have absolutely no difficulty looking at a mercury thermometer at 9:00 and determining that the temperature is 30 degrees. This observation is a conversion from a continuous "signal" (temperature as a function of continuous time) to a discrete sample - a snapshot if you like. (Incidentally, likely this observation has involved quantization - we have likely rounded to 30 degrees - but this fact is not essential to sampling itself.)

What may make the mathematics cumbersome is the association of sampling with the corresponding Fourier transform ideas. We generally have some notion of the "spectrum" of a signal in terms of its Continuous Time Fourier Transform (CTFT) and we would like to know what the spectrum looks like after the signal is sampled. In the simplest terms, what happens is that the original spectrum of the continuous-time signal is replicated about each integer multiple of the sampling frequency. Thus our procedure when we recognize that we have a sampling problem is:

1) Draw a straight line and mark one point as zero frequency.

2) Mark off some integer multiples of the sampling frequency. For example, $-f_S$, 0, $f_S$, and $2f_S$ will usually suffice.

3) Identify the spectrum of the original continuous-time signal, and sketch it centered about zero.

4) Replicate this sketch about all multiples of the sampling frequency. Worry about getting the frequency scale correct, but don't worry about the amplitudes until and if it becomes absolutely necessary.

5) Study the resulting situation.

For example, Fig. 1a shows a case where we have a triangular shaped spectrum with a maximum frequency of $f_{max} = 0.4\ f_s$. Note that the spectral replicas do not overlap each other anywhere. Fig. 1b on the other hand shows the case where $f_{max} = 0.6\ f_s$, and in this case, the spectral replicas do overlap in part - the phenomenon known as "aliasing." This observation alone leads us to the most basic understanding of the "sampling theorem" - that the bandwidth must be limited to less than half the sampling frequency. In such a case, we expect to be able to recover the signal exactly from its samples by using a low-pass filter with a cutoff at $f_s/2$.

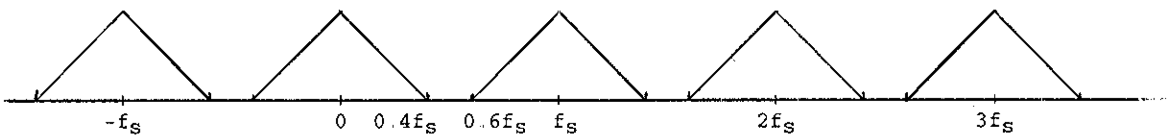



Fig. 1a Ordinary sampling - the original spectrum, a triangular shape between $-0.4f_s$ and $0.4f_s$, is replicated about integer multiples of $f_s$. The original spectrum can be recovered with a reasonable low-pass filter that is relatively flat from 0 to $0.4f_s$, and which rolls off to negligible gain at $0.6f_s$.

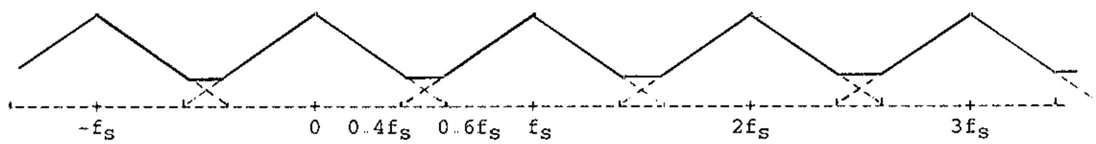



Fig. 1b Here the original spectrum is wider than the case of Fig. 1a, extending up to $0.6\ f_s$. This original spectrum is still replicated about all integer multiples of $f_s$. Now there is overlap of replicas (aliasing) between $0.4f_s$ and $0.6f_s$. If the original spectrum were low-pass filtered to $0.5f_s$ prior to sampling (a "house" shape rather than a triangle), there would be no aliasing, but the information between $0.5f_s$ and $0.6f_s$ would of course be lost. The aliasing of Fig. 1b can not be undone, but the corrupted portion could be removed with a low-pass filter with cutoff $0.4f_s$. In this case, the information in the original spectrum between $0.4f_s$ and $0.6f_s$ would be lost.

On the other hand, in some instances we can be much more flexible in our interpretation of the sampling situation. For example, the spectrum may be bandpass-like, in which case, even when the frequencies being sampled are greater than $f_s/2$ (e.g., Fig 1c), we may well not have any overlap of replicas, and we could recover the signal just fine (in this case, with a bandpass reconstruction filter). Also, for signals with bandwidths much less than $f_s/2$, we often have considerable latitude in our choice of reconstruction filter. Finally, it is not even necessary that samples are uniformly spaced in time (as long as the average sampling rate is high enough). Of course, in the majority of cases, the signals we sample are low-pass in nature and sampled uniformly in time.

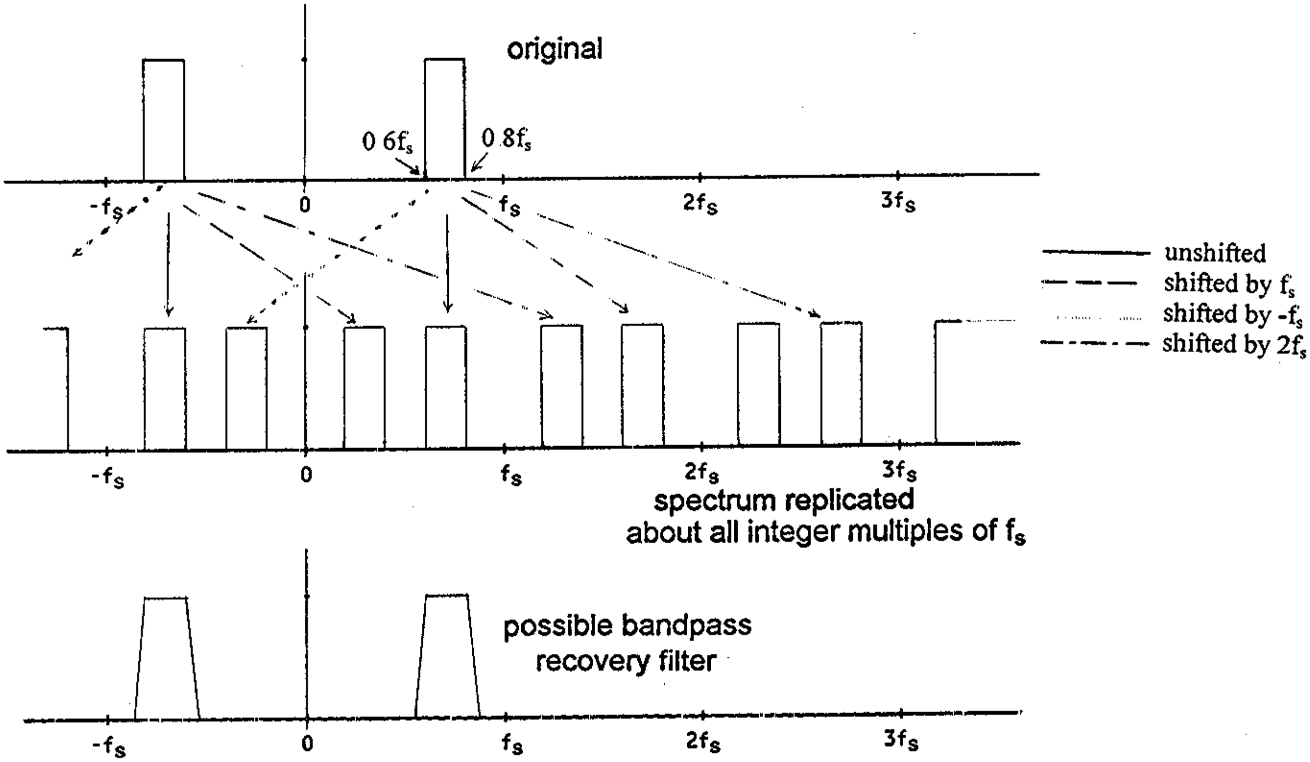



Fig. 1c Here the original spectrum actually exceeds $0.5f_S$ everywhere, but has a finite one-sided bandwidth that is less than $0.5f_S$. In such a case, it is possible to recover the original spectrum completely. In this particular case, this can be seen to be very simple, since there is no overlap, and a bandpass filter is simply employed. If there is overlap, a more complicated procedure is required.

Many practicing engineers may have no conceptual problems with sampling - recognizing that specific electrical devices are capable of encoding from continuous signals to "point samples" (sequences of numbers) in a wholly useful manner. Further, these same engineers may well achieve success in both sampling, and reconstruction from samples, based on the simple (visual) procedures suggested above, particularly as they have developed a good measure of engineering intuition. Others may still desire to explore the mathematical tools, but even they are advised to first appreciate the visualizations that are possible.

# 2. SAMPLING MATHEMATICS

## 2a. IMPULSE SAMPLING: SAMPLING WITH DELTA-TRAINS

One popular (although also problematic) procedure for demonstrating the spectral replication property that results from sampling is to invoke the so-called "Shah Function" which is a periodic train of Dirac delta functions. (Accordingly, the mathematical baggage that accompanies the Dirac delta's comes along here.) The point of view is to consider the samples to be the result of multiplying the continuous time signal by the time-domain delta-train. Then, we look in the frequency domain for convolution of the original spectrum with the Fourier transform of a delta-train This Fourier transform of the time domain delta-train is a delta-train in frequency. If the spacing of the impulses in time is T, the spacing in frequency is 1/T Hz, or $2\pi/T$ radians/second. We thus easily see convolution as handing us the replication picture with frequency spacing being the sampling frequency, with scaling constants to be determined. The above may be all some readers may need to know about this approach.

Some intuitive measure of the understanding of the delta-train relationship can be afforded by recalling the Fourier series of a pulse train. Recall that the Fourier series coefficients are given by:

$$c(k) = (1/P) \int_{-P/2}^{P/2} f(t)\, e^{-j(2\pi/P)kt}\, dt \qquad (1)$$

where P is the period of a periodic function f(t). [Alternatively we simply take the CTFT of one cycle of f(t) and sample it at frequencies $\Omega = (2\pi k/P)$, dividing the results by P.] In either case, for a pulse of width $\tau$, centered about t=0, repeating with period T, we get:

$$c(k) = (\tau/T)\,[\sin(\pi k\tau/T) / (\pi k\tau/T)] \qquad (2)$$

As $\tau$ approaches 0, we see two things happening. The term in [ ], the sinc function, flattens out to a constant 1. But at the same time, the $(\tau/T)$ factor goes to zero. Fixing this situation by multiplying the original pulse height by $1/\tau$, we end up supposing that the spectrum is c(k)=1/T as $\tau$ approaches 0. In multiplying the original pulse train by $1/\tau$, we maintain an area of 1 under each pulse even as the width shrinks. At the same time, we need to recognize that the c(k) themselves are to be considered delta-trains because they are samples (in frequency) of the exact same nature as the ones we have just taken in time. Thus we find an amplitude scaling of 1/T, and a corresponding dimensional irregularity, as we go from the original CTFT spectrum to the sampling replicas. In order to correspond exactly to the conventional DTFT, we would need to normalize the frequency axis so that the sampling frequency is $2\pi$ (without any physical dimensions), usually by thinking of T as having a dimensionless value of 1.

## 2b. DIRECTLY RELATING THE CTFT AND THE DTFT

The Continuous Time Fourier Transform (CTFT), $X_a(\Omega)$, of a signal $x_a(t)$ is given by:

$$X_a(\Omega) = \int_{-\infty}^{\infty} x_a(t)\, e^{-j\Omega t}\, dt \qquad \textbf{[CTFT]} \qquad (3a)$$

with inverse:

$$x_a(t) = (1/2\pi) \int_{-\infty}^{\infty} X_a(\Omega)\, e^{j\Omega t}\, d\Omega \qquad \textbf{[Inverse CTFT]} \qquad (3b)$$

while the usual form of the Discrete Time Fourier Transform (DTFT), $X(e^{j\omega})$, of a discrete time sequence x(n) is:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jn\omega} \qquad \textbf{[Usual DTFT]} \qquad (4a)$$

with inverse:

$$x(n) = (1/2\pi) \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{jn\omega}\, d\omega \qquad \textbf{[Usual Inverse DTFT]} \qquad (4b)$$

Here the frequency $\omega$ is the "normalized" version of frequency, which is considered to be dimensionless. This can be related to the physical frequencies $\Omega$ (radians/second) or f (Hz, or cycles/second) as:

$$\omega = 2\pi(\Omega/\Omega_s) = 2\pi(f/f_s) = 2\pi fT = \Omega T \qquad (5)$$

where $\Omega_s$ and $f_s$ are the corresponding sampling frequencies, and $T=1/f_s$ is the sampling period. Since the CTFT is expressed in terms of physical frequency, it will be very useful to also represent the DTFT and its inverse in terms of $\Omega$ instead of $\omega$. That is, we insert $\Omega T$ for $\omega$, and this results in the equations:

$$X(e^{j\Omega T}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jn\Omega T}$$ **[Physical DTFT]** (6a)

$$x(n) = (T/2\pi) \int_{-\pi/T}^{\pi/T} X(e^{j\Omega T})\, e^{jn\Omega T}\, d\Omega$$ **[Physical Inverse DTFT]** (6b)

where the limits on the integral from $\omega = -\pi$ to $\pi$ become $-\pi/T$ to $\pi/T$ in terms of $\omega = \Omega T$, and the T outside the integral results from $d\omega = T d\Omega$, all the rest of the reformation being simple substitution.

At this point we want to look at the case $x(n) = x_a(t=nT)$ where T is the sampling interval. This procedure (which we find in 1975 era DSP textbooks) will lead us to the same result of spectral replication that we find with impulse sampling. We begin with equation (3b), the inverse CTFT, which is true for all t, and therefore certainly for t=nT

$$x_a(t=nT) = (1/2\pi) \int_{-\infty}^{\infty} X_a(\Omega)\, e^{j\Omega nT}\, d\Omega \tag{7a}$$

The integral over infinite limits in equation (3b) can be represented as a infinite sum over finite segments of length $2\pi/T$ as:

$$x_a(t=nT) = (1/2\pi) \sum_{m=-\infty}^{\infty} \int_{\Omega=(2m-1)\pi/T}^{\Omega=(2m+1)\pi/T} X_a(\Omega)\, e^{j\Omega nT}\, d\Omega \tag{7b}$$

By a change of variable, $\Omega = \Omega' - 2m\pi/T$, we have:

$$x_a(t=nT) = (1/2\pi) \sum_{m=-\infty}^{\infty} \int_{\Omega'=-\pi/T}^{\Omega'=\pi/T} X_a(\Omega')\, e^{j\Omega' nT}\, d\Omega' \tag{7c}$$

or (since $d\Omega' = d\Omega$):

$$x_a(t=nT) = (1/2\pi) \sum_{m=-\infty}^{\infty} \int_{-\pi/T}^{\pi/T} X_a(\Omega + 2\pi m/T)\, e^{j(\Omega+2\pi m/T)nT}\, d\Omega \tag{7d}$$

Recognizing that $e^{j2\pi mn}=1$ for all integers m and n, and multiplying by T/T=1 we have:

$$x_a(t=nT) = (T/2\pi) \sum_{m=-\infty}^{\infty} (1/T) \int_{-\pi/T}^{\pi/T} X_a(\Omega + 2\pi m/T)\, e^{j\Omega nT}\, d\Omega \qquad (7e)$$

Reversing the order of summation and integration:

$$x_a(t=nT) = (T/2\pi) \int_{-\pi/T}^{\pi/T} [\,(1/T) \sum_{m=-\infty}^{\infty} X_a(\Omega + 2\pi m/T)]\; e^{j\Omega nT}\, d\Omega \qquad (7f)$$

This integrand is exactly of the form of equation (6b), the physical inverse DTFT, with:

$$X(e^{j\Omega T}) = [\,(1/T) \sum_{m=-\infty}^{\infty} X_a(\Omega + 2\pi m/T)] \qquad (8a)$$

Noting that $2\pi/T = 2\pi f_s = \Omega_s$, the sampling frequency in radians/second:

$$X(e^{j\Omega T}) = [\,(1/T) \sum_{m=-\infty}^{\infty} X_a(\Omega + m\Omega_s)] \qquad (8b)$$

which clearly shows spectral replications, spaced at intervals of the sampling frequency, and scaled by 1/T. In terms of $\omega$:

$$X(e^{j\omega}) = (1/T) \sum_{m=-\infty}^{\infty} X_a(\omega/T + 2\pi m/T) \qquad (9a)$$

or:

$$X(e^{j\omega}) = (1/T) \sum_{m=-\infty}^{\infty} X_a[\,(\omega + 2\pi m)/T\,] \qquad (9b)$$

showing spectral replicas at spacing $2\pi$.

# 2c. RECOVERY WITH SINC INTERPOLATION

## 2c-1 Basics

Given a set of discrete samples, how do we recover the original, continuous-time signal? The easiest way of answering this question is to observe that an appropriate low-pass (or possibly other type) filter will recover the original spectrum, i.e., reject the sampling-generated replicas (Fig. 2). Then we have only to recognize the uniqueness property of the Fourier Transform to claim the recovery of the original time domain signal.

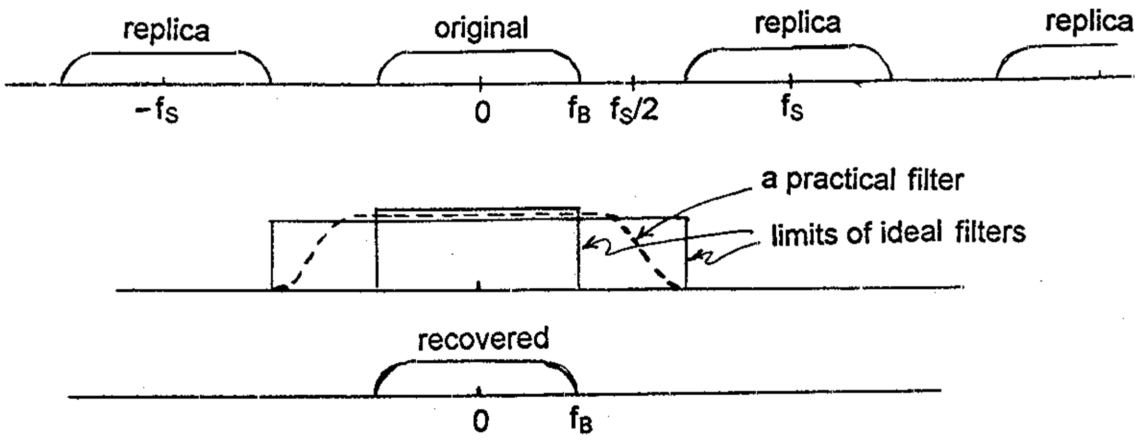



Fig. 2 A Variety of filters are possible for reconstruction

There are two situations in which this question of recovering from samples may be asked. In the first situation, we know that the samples were obtained from a continuous-time waveform, and our goal is to recover that waveform (for example, a digital recording of a musical performance). In the second situation, the samples are handed to us and we are asked to "make what we can" of them. Neither of these cases makes sense until we come to terms with some realistic notion of a bandwidth associated with the samples.

It is not infrequently supposed that the bandwidth is automatically half the sampling frequency. This supposition is perhaps attractive in that the mathematics for this special case gives a simple easy-to-sketch picture, and further, this view can be a consequence of assuming (1) that the sampling theorem was obeyed and (2) that we are being conservative and assuming that the worst case is being approached. Corollary to this assumption would be the assurance that a signal bandlimited to a values $f_B$ less than $f_S/2$ is also bandlimited to $f_S/2$. The major problem with this assumption that the bandwidth is $f_S/2$ is the resulting requirement of a reconstruction filter that is ideal. Note that in the case where we know the samples were obtained from a continuous-time signal we have an excellent chance of knowing that the band-width was less than $f_S/2$, and there is always the need to achieve reconstruction with a practical low-pass filter.

## 2c-2 Ideal Low-Pass

Although we have no chance of ever finding an ideal low-pass filter to install in our projects, we nonetheless make good use of ideal low-pass filters as mathematical objects to indicate general results. In Fig. 2, we show a practical, non-ideal, gradual cutoff low-pass which is sufficiently flat in the band below $f_B$ and is of negligible gain above $f_S$-$f_B$. We could of course use any ideal low-pass filter that has a cutoff $f_C$, where $f_C$ is above $f_B$ and below $f_S$-$f_B$.

The filtering process is a frequency-domain multiplication. Correspondingly, in the time domain we want to convolve the signal that is input to the filter with the impulse response of the low-pass filter.

$$f(t) = (1/2\pi) \int_{-\infty}^{\infty} F(\Omega)\, e^{j\Omega t}\, d\Omega \qquad \textbf{[Inverse Continuous-Time Fourier Transform]} \qquad (10)$$

Here $F(\Omega)$, the filter's frequency response is rectangular: it vanishes outside of the interval $-\Omega_C$ to $+\Omega_C$, where $\Omega_C=2\pi f_C$ is the cutoff frequency in radians/second. Inside the passband, we will take the frequency response to be T. This value of T, instead of just 1, is needed to make up for the 1/T that we found scaling the replicas in equations (9a) and (9b). Equation (10) is easily integrated to:

$$f(t) = (\Omega_C T/\pi)\,[\,\sin(\Omega_C t) / \Omega_C t\,] = (2f_C T)\,[\,\sin(2\pi f_C t) / 2\pi f_C t\,] \qquad (11)$$

which is in the form of a familiar sinc function, the expected result of transforming a rectangle. At this point we note the simplicity that results from choosing $f_C=f_S/2 = 1/2T$. This means that equation (11) is zero when t is a multiples of T, except at t=0 where equation (11) becomes 1.

To more fully appreciate this simplicity, note that the convolution procedure needed here is equivalent to summing a set of weighted and displaced version of equation (11). That is:

$$x(t) = (2f_C T) \sum_{n=-\infty}^{\infty} x(n)\ \{\,\sin[2\pi f_C(t-nT] / 2\pi f_C(t-nT)\,\} \qquad (12a)$$

which becomes, for $f_C=f_S/2$:

$$x(t) = \sum_{n=-\infty}^{\infty} x(n)\ \{\ \sin[(\pi/T)(t-nT)]\ /\ (\pi/T)(t-nT)\ \} \qquad (12b)$$

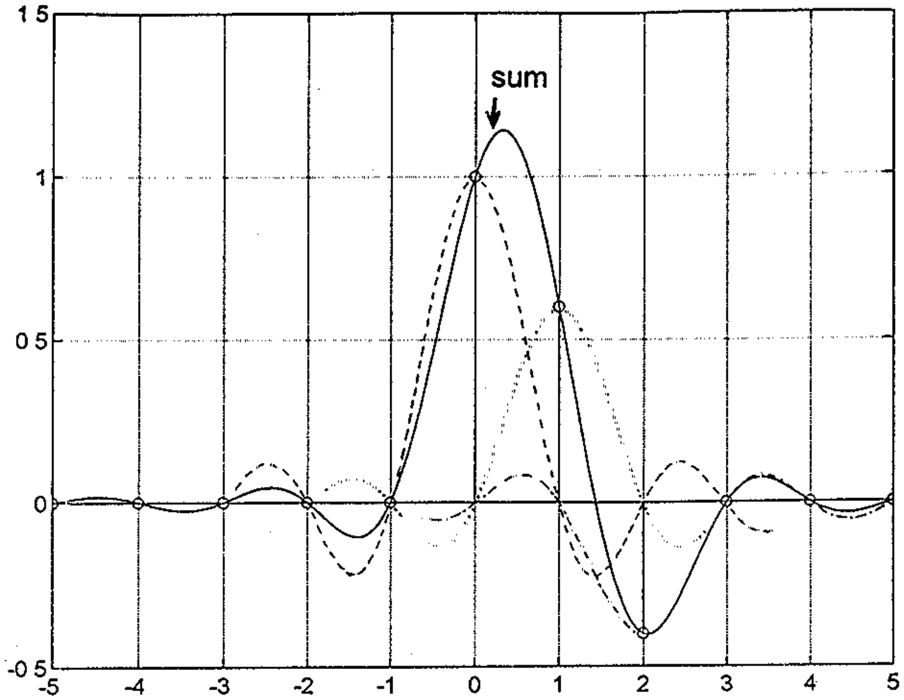



Fig. 3 A "textbook example" where the non-zero samples are convolved with a sinc function (corresponding to an ideal low-pass with cutoff equal to 0.5.

Fig. 3 shows an example of this reconstruction. Here we assume that there are only three non-zero samples, x(0)=1, x(1)=0.6, and x(2)=-0.4. These are shown in terms of the individual weighted sinc contributions, and the sum of the three. Note that for this special case, the reconstructed curve goes exactly through each of the original points. This statement includes the points not specified (thus zero by default) - it goes exactly through these zeros. Yet note well that it is not zero except at these integer points. The recovered curve is of course a perfectly good waveform bandlimited to $0.5f_S$. Accordingly, we could resample this curve at the same sampling frequency $f_S$=1 with a different initial starting point. The resampling could be done at (integers+1/4) for example, in which case we would get non-zero samples for all n. Thus while we may get the impression (from Fig. 3) that the situation here involves a signal that is nearly always zero (is non-zero only for three discrete samples), the actual signal is, in general, always non-zero, either as a continuous-time signal, or as samples. Accordingly, Fig. 3 is a "textbook example" but must be understood to be a doubly special case (special bandwidth, special initial timing).

## 2c-3 Bandwidth Assumptions

We can not escape the issue of bandwidth which we have introduced above. First of all, we need to deal with signals that have bandwidths less than $f_S/2$, since we are going to use real filters in association with our sampling. In fact, it is this issue of using real filters that provides us with a simple (perhaps even sufficient) manner of dealing with bandwidth ambiguities.

In this reality-based view, we know that we will often have need of two, real, continuous-time low-pass filters. The first of these is the so-called "input guard" or "anti-aliasing" filter placed between the signal to be sampled and the actual sampling device. In a useful sense, this anti-aliasing filter defines the bandwidth. It not only tells us the general low-pass cutoff frequency (the bandwidth) but also specifies the rejection properties above the cutoff. We choose this so that we feel that, for practical purposes, for expected input signals, there will be insignificant spectral energy above $f_S/2$. In a conservative design procedure, the nominal cutoff frequency is chosen to be below $f_S/2$, allowing room for a reasonable roll-off by $f_S/2$. For example, we might look to use a 3.7 kHz low-pass cutoff for an 8 kHz sampling rate (typical of speech).

With the bandwidth now provisionally described in terms of an input guard filter, it makes good sense to consider our second filter, a reconstruction filter (also called a "smoothing filter" or an "anti-imaging" filter) that has the same bandwidth; perhaps even as being identical in design and realization to the anti-aliasing filter. This bandwidth-as-defined-by-filters view is useful. In cases where we have only the reconstruction filter (for example, the digital synthesis of musical sounds), we can still think of bandwidth in terms of the reconstruction filter's cutoff.

At some point however, we may wish to obtain a more rigorous understanding. We know for example that if we have a signal known to be bandlimited to $0.4f_S$, we can sample it at $f_S$, and then we could recover it with a variety of filter shapes and a range of cutoff frequencies for the filter that have satisfactory (flat) passband properties, and satisfactory rejection by $0.6f_S$. How would we demonstrate this? In fact, how do we get a test signal that is bandlimited to $0.4f_S$?

Taking a clue from our discussion above relating to the definition of bandwidth in terms of a filter from which the signal emerges, we see that we can get a signal bandlimited to $0.4f_S$ by using a sum of appropriate sinc functions. In fact, we "construct" this signal pretty much as we "reconstruct" from samples, using equation (12a). One difference is that in constructing a bandlimited signal, we don't even need equally spaced samples, or even samples as such. Essentially anything could be convolved with the sinc of equation (12a) to get a test signal. The sinc itself could be used, but we might feel more comfortable with something less special.

Now for the interesting part. Note that when we now sample this signal at $f_S$, we expect in general to get an infinite number of non-zero samples. This is even true for a bandwidth of $0.5f_S$ with a general starting time (as we noted above). This suggests that we really do need to sum the reconstruction equation (12a) over an infinite number of terms to get the right answer. But two things work for us, both which involve the roll-off of 1/t that is in the denominator of the sinc function. First, because of this roll-off, the sinc contribution due to samples that are far from the current region of interest will be minor. Secondly, if we have constructed our test signal from samples (i.e., sincs that are centered on these samples) that are inside or near the current region of interest, the samples taken far from the region of interest are small to begin with. Put another way, the sinc roll-off makes samples smaller and smaller as we go out from the center, and further makes the contribution from these far away samples even smaller as we come back. All and all, on ordinary plots, we can see perfect or near perfect recovery even from a finite, relatively small number of summed sincs.

In doing this experiment, it may be at first disconcerting that a sinc, convolved with a sample, will not go through that sample for cases where the corresponding filter's cutoff is not $0.5f_S$. This is a matter of the gain and change of the spacing of zeros in equation (12a). Of course, these sincs will not go through zero at the other sampling points either. What we do know is that the sum of all the sincs involved will work.

One last point of interest is that once we have filtered with a low-pass filter with a cutoff at $0.4f_S$, for example, we could follow this with a low-pass filter with a cutoff at $0.45f_S$, for example, (or anything that is flat on 0 to $0.4f_S$) and we expect no change, as is obvious from the frequency domain picture (multiplying two rectangles gives the smaller rectangle). In consequence, it must be true that the convolution of two sinc functions gives the one sinc of the pair that corresponds to the narrowest rectangle, the sinc with the more widely spaced zero crossings. This strange result is helpful when we consider a signal constructed at one bandwidth being reconstructed at another.

Fig. 4 (a - f) shows examples of experiments that can be done starting with these ideas. Three somewhat different but related experiments are found here in pairs of figures: 4a and 4b, 4c and 4d, 4e and 4f The captions that accompany the figures contain many details.

Fig. 4a and Fig. 4b relate to the construction of a signal with a bandwidth of 0.3, which is then sampled at 1, and reconstructed with a bandwidth of 0.7. All is fair here -and we might expect a perfect reconstruction, but this would require an infinite number of samples. With only the nine samples shown, the reconstruction is imperfect, but would improve if me kept more samples. For example, if we had kept samples from say -25 to +25, and reconstructed from these 51 samples, the region shown from -4 to +4 would look virtually perfect, although we would expect there to be at least some noticeable truncation errors around -25 and +25 now.

Injudicious choice of filter bandwidth leads to two common errors, aliasing in Fig. 4c, and an attempted reconstruction that ends up contaminated by part of a higher frequency image in Fig. 4d. These look similar, but note well that the output in Fig. 4c is "lower in frequency," and the output in Fig. 4d is "higher in frequency," relative to the input. The first is an anti-aliasing failure, and the second is an anti-imaging failure. Neither result is correct or acceptable.

In Fig. 4e and Fig. 4f, we show two examples of bandlimited signals that do go through original prescribed points. In Fig. 4a we saw that using three samples (0,1), (1,2), and (2,-1) as input to a filter with a cutoff of 0.3 did not result in a signal that went through these points. Is it possible to find a waveform, bandlimited to 0.3, that does go through these points? The answer is yes, and there are an infinite number of such waveforms, of which we show two. In these cases, we find, as unknowns, the values of initial samples at particular times, which, if passed through a specified low-pass filter, go through the second set of prescribed points (knowns). This is better thought of as a sinc expansion, and the solution is a mater of solving N equations in N unknowns.

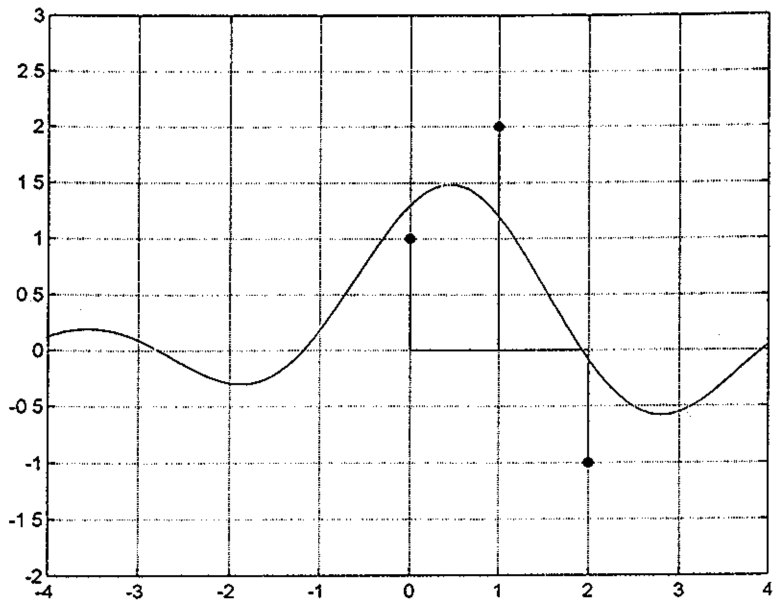


Fig. 4a Three samples [solid dots at (0,1), (1,2), and (2,-1)] pass through an ideal low-pass with cutoff 0.3, using equation (12a). The waveform is bandlimited to 0.3. It does not go through the original points (it need not). No scaling and/or shifting of time and/or amplitude will make it go through these points. (See also Fig. 4e). It is a perfectly good test waveform.

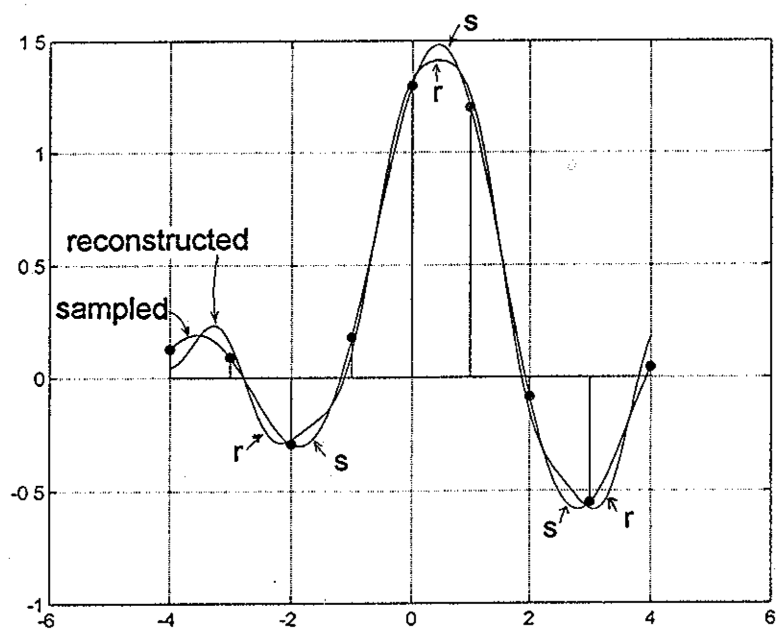



Fig. 4b The test waveform of Fig. 4a is sampled at integers -4 to +4 (solid dots). Using only these nine samples, the reconstruction shown is achieved, using a low-pass cutoff of 0.7. It is better near the center, and would be better still if more samples were used in the reconstruction.

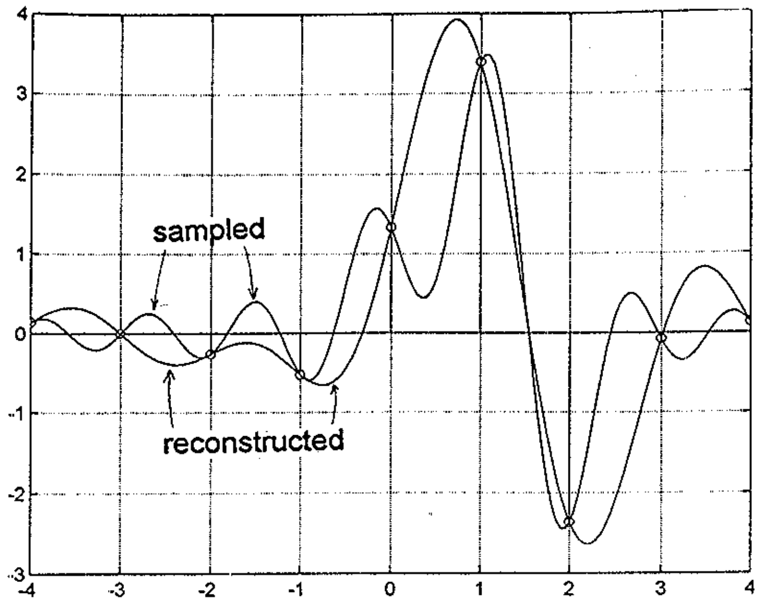



Fig. 4c Here the sample points [(0,1),(1,2), and (2,-1)] are constructed with a low-pass with cutoff 0.85, then sampled at 1, and reconstructed with a cutoff of 0.5. The reconstruction shows aliasing (lower frequency output) because the input bandwidth exceeds 0.5. This is a failing of the anti-aliasing (guard) filter at the input.

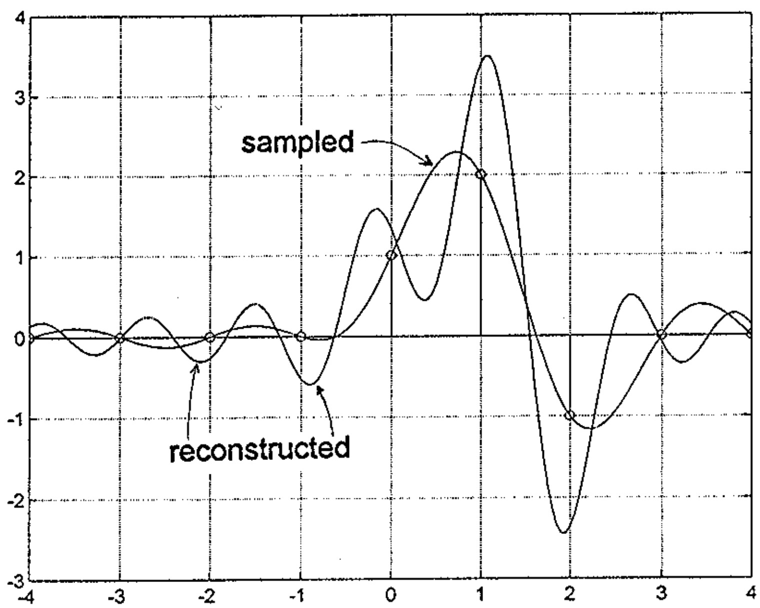



Fig. 4d Here the samples [(0,1), (1,2), and (2,-1)] are constructed with a bandwidth of 0.5, sampled at a rate 1 (no problem), but then reconstructed with a filter with a cutoff of 0.85. The reconstruction shows higher frequency components because a portion of a sampling replica (0.5 to 0.85) is included. This is a failure of the output anti-imaging (reconstruction, smoothing) filter.

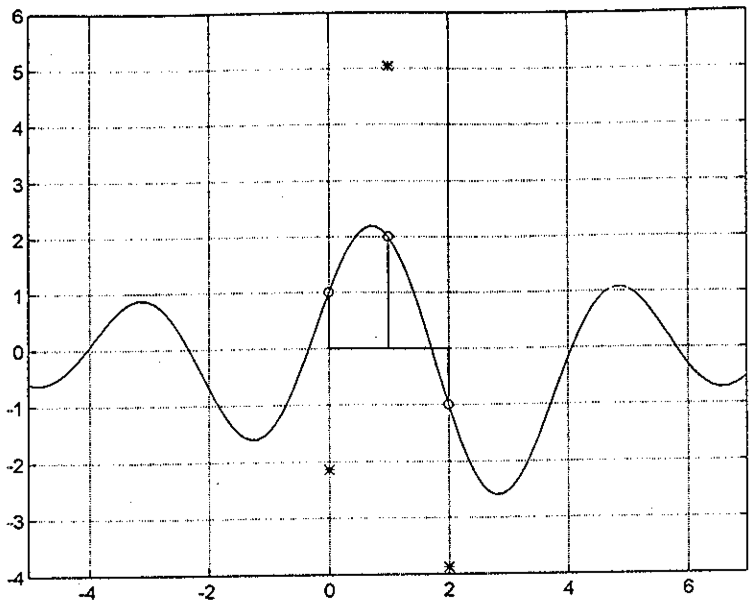



Fig. 4e There exist samples at t=0,1, and 2, which when passed through an ideal low-pass with cutoff 0.3 will have an output that goes through (0,1), (1,2), and (2,-1), and these are shown by the (*'s). Finding these is just a matter of solving three equations in three unknowns, starting with the sinc expansion. Compare to Fig. 4a.

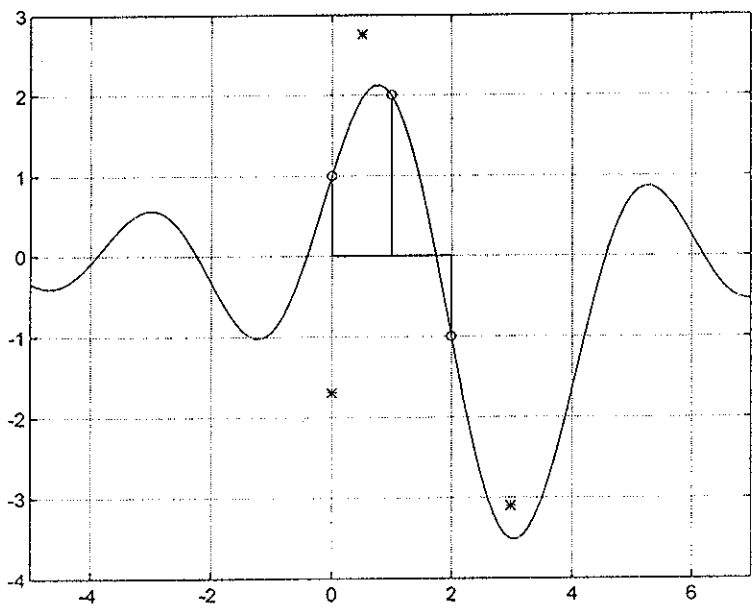



Fig. 4f The waveform, bandlimited to 0.3, that passes through (0,1), (1,2) and (2,-1) as seen in Fig. 4e is not unique, if the input samples can be located at other times. Here we show samples at times 0, 0.5, and 3.

## 2d. SAMPLE-AND-HOLD AS A STEP IN RECONSTRUCTION

One thing that may bother a student of DSP is that the samples we deal with supposedly have zero width. Moreover, the spectrum of these samples is replicated an infinite number of times Perhaps we could argue that there is a trade-off: having an infinite amount of energy, but only having it for zero time, somehow compensates. But it is more to the point to recognize that the signals we deal with as mathematical sequences of numbers have no automatic corresponding physical reality. Physical reality is imposed on a signal only when we convert the signals from a mathematical to an electrical form, and this is almost always done with a D/A converter. Specifically, the D/A converter outputs a voltage that corresponds to a particular number, and it holds this voltage until such time as the digital input changes. Thus we are dealing with, and frequently think in terms of, a sample-and-hold. Equivalently, our sequences represent piecewise constant functions (stepped approximation) when physically realized

It is intuitive that a piecewise constant situation is a step in the direction of moving from a discrete to a continuous signal. This is particularly evident as the step length is made very small (sampling rate is high). In this situation, the "samples" clearly have non-zero width, and the energy in the signal is clearly finite It follows that, due to the S&H, the spectral images must somehow be rolling off at high frequencies. That is, the inherent sample-and-hold action is somehow some form of low-pass filtering. This "voluntary" contribution to the recovery simplifies some things, and complicates others.

Specifically, the sample-and-hold (also called a zero-order hold) can be regarded as the convolution of a sequence of samples with a rectangular pulse of width T, where T is the sampling time. Mathematically, we need to do this by making our samples into Dirac delta functions that are weighted by the samples, and then doing convolution as integration. However, the simple picture of Fig. 5 gives us the essential idea

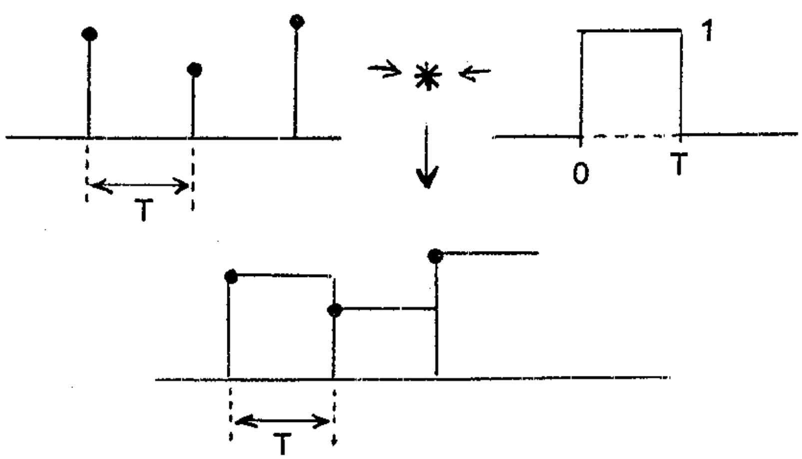



Fig. 5 Sample-and-Hold as convolution of point samples with a rectangle

As usual, we will find it convenient to look at multiplication in the frequency domain instead of this time-domain convolution. The Continuous-Time Fourier Transform (CTFT) of the rectangular pulse is of course:

$$H_H(\Omega) = \int_{-\infty}^{\infty} h_H(t)e^{-j\Omega t}\,dt = \int_{0}^{T} e^{-j\Omega t}\,dt$$

$$= Te^{-j\Omega T/2}\ \sin(\pi T/2)/(\pi T/2) = (1/f_S)e^{-j\pi f/f_S}\ \sin(\pi f/f_S)/(\pi f/f_S) \qquad (13a)$$

This frequency response, like that of the ideal low-pass of equation (12a), has a scaling of T that makes up for the 1/T scaling of the sampling replicas. Here we have also shown the exponential term $e^{-j\Omega T/2}$ which is due to the hold being extended **causally** here (a delay of T/2 as seen in the frequency domain). But what is most important is the sinc roll-off:

$$|H_H(\Omega)| \propto |\sin(\pi f/f_S)/(\pi f/f_S)| \qquad (13b)$$

This roll-off is shown in Fig. 6. If we assume, for example, that the spectrum corresponding to point samples is significant for frequencies from 0 to $0.4f_S$, we can see that the S&H roll-off causes a significant roll-off in the band, reaching a level of 0.7568 (-2.42db) at $0.4f_S$. Whether or not this is important depends on the particular

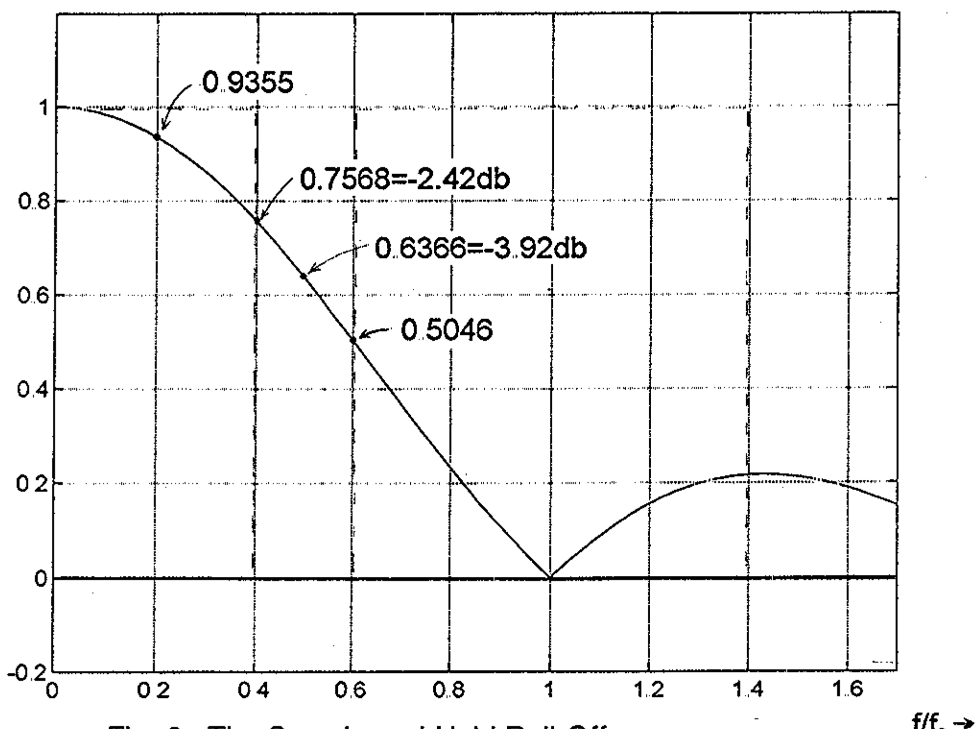



Fig. 6 The Sample-and-Hold Roll-Off

application. We might suppose that it would not be a problem with telephone speech, but could be for music. If it is a problem, reasonable corrections for this roll-off are known, and could be made part of the digital processing prior to D/A conversion, or could be part of the analog reconstruction filter. Another approach would be to shorten the hold time, making it a fraction of T (see Section 3a).

We note as well that the sinc roll-off helps us remove replicas, since the sinc roll-off goes to zero at multiples of the sampling frequency. From Fig. 6, however, it is clear that it is not all that effective for the first replica at least, rejecting only to about 1/2 at $0.6f_S$.

All and all, we must have a S&H in order to output physical energy for a real-world signal. Given this fact, we have to deal with the roll-off correction when necessary, and at least it helps, rather than hinders, the removal of replicas.

## 2e. OVERSAMPLING

It is evident that we can not obtain ideal low-pass filters for recovering signals from samples, nor should we necessarily expect the actual realization of filters that are satisfactory (for practical purposes), to be trivial. While we note that the process of designing a recovery filter should end up with a device that is inherently analog (continuous time), doing the entire job as a traditional analog "active filter" is no longer the only approach. In the past, we designed analog filters to make it possible to utilize digital filters. Today we design digital filters to make the residual analog filtering requirements much easier. This is one application of what we call "oversampling," which is in turn one branch of the multi-rate DSP art. (Reduction of quantization noise and a more linear phase response are additional reasons for using an oversampoling approach.)

The problem of designing a satisfactory recovery filter is a matter of obtaining a flat passband and a roll-off rate that is fast enough to get the response down to a negligible value before a spectral replica is encountered. For example, if a signal has a bandwidth of 4.7 kHz and is sampled at 10 kHz, then we need to use a low-pass recovery filter that is sufficiently flat up to 4.7 kHz, but which then gets down to a negligible gain by 5.3 kHz, a transition region of only 600 Hz. In practice, this might lead to the need for something like a 10th to 16th-order analog active filter, which can be very difficult to build.

Fig 7 is a sketch of an example spectrum corresponding to an original signal having a bandwidth that is flat between -1 and +1 Further, the signal is sampled at a sampling frequency of 3, so for the frequency range shown, we see replicas centered about 3 and about 6. We give these rectangles an amplitude of 0.8 to avoid a cluttered diagram since we have also plotted some filter shapes, which are our real interest here.

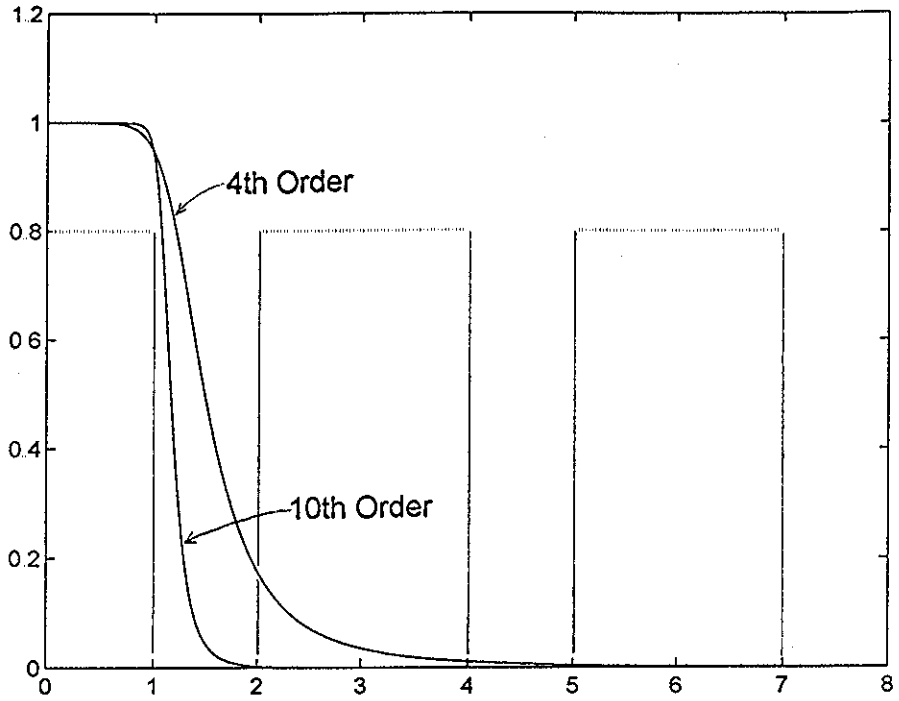



Fig. 7 Two Possible Analog Low-Pass Recovery Filters

Two analog filters are shown. The nominal job of these filters it is to pass frequencies from 0 to 1, but to then get down to negligible gain by a frequency of 2, thus removing the replicas between 2 and 4, 5 and 7, and so on (i.e., all replicas). The first filter is a 4th-order analog Butterworth low-pass with a -3db cutoff set to a frequency of 1.3. It is reasonable to contend that this filter is not satisfactory since it would pass a considerable portion of the replica between 2 and 4, and is not particularly flat in the passband. The second filter is an analog 10th-order Butterworth low-pass with a -3db cutoff at 1.1. This filter is apparently more satisfactory, rejecting well by the frequency of 2, and being flatter in the passband, and would be our choice of the two. The practical difference however is that the 4th-order filter could be easily constructed with two op-amps and a dozen passive components (8 resistors and 4 capacitors) with tolerances no better than 5% (i.e., all inexpensive easily obtained components). The 10th-order filter on the other hand would require 5 op-amps, 10 capacitors, and 20 resistors, and these would need to be of a tolerance of perhaps 2% or 1%, and might even require hand trimming. This "sensitivity" issue with active filters is well-understood.

With oversampling, we would try (for example) to somehow remove the spectral replica that is between 2 and 4 prior to employing the analog filter. If we can do this, then the fourth-order filter could likely be used. The point to note here is that if we remove this replica centered at 3, (and also remove the replicas centered at 9, 15, etc., while keeping those centered at 6, 12, 18, etc.) then the resulting spectrum is exactly what we would have had if the same original signal had been sampled at 6 rather than at 3. Such a development can be considered "oversampled" because achieving the equivalent of a sampling frequency of 6 would be a much higher sampling rate than the minimum required. (Here the minimum sampling frequency would have been 2. We actually chose 3 to allow room for the cutoff region of a practical filter. We would need a special reason to go to 6, and here we have it.)

Of course, in most cases we don't expect to achieve the removal of half the spectral replicas by going back and sampling at a higher rate. But we also conclude that the samples we would have taken are not totally unavailable to us either. This is because, with the proper effort (a 25th-order filter perhaps), we can reasonably return to continuous time, thus obtaining the time signal at all times, and of course therefore, at all intervals of 1/6. Our actual effort is not unlike that suggested: using a high-order filter. The difference is that it will be a digital filter. There are three important points concerning this approach: (1) we digitally process the samples we already have to obtain the missing samples, (2) we can implement high-order digital filters easier than we can analog filters with similar performance, and (3) we are dealing with a periodic frequency response, which is slightly less convenient than the analog "get down and stay down" response.

While we will shortly return to this filtering (frequency domain) approach, we also find it useful to think in terms of the time domain. What we are looking for is a means of obtaining "stand-in" replacements for samples we failed to take - at least a reasonably good estimate of the missing samples. We naturally think in terms of some type of interpolation, and the simplest case (other than a hold) is linear interpolation. This means that we estimate a missing sample between two known samples as the average of the two known samples. In terms of a digital filtering operation, approached through time-domain thinking, we would first place zeros midway between the original samples, thereby increasing the sampling rate from 3 to 6 (but not changing the physical spectrum - See Fig. 14b). This zero-padded sequence is then convolved with a three-tap FIR filter that has impulse response values 1/2, 1, and 1/2. For convenience take:

$$\begin{aligned} h(-1) &= 1/2 \\ h(0) &= 1 \\ h(1) &= 1/2 \end{aligned} \qquad (14)$$

from which we get the frequency response from the DTFT, equation (6a) as:

$$H(f) = 1 + \cos(2\pi f/6) \qquad (15)$$

This filter (frequency response magnitude is shown in Fig. 8c) has a double zero at z=-1 in the z-plane, and is a crude low-pass. Accordingly, while the procedure was initiated as time-domain interpolation, the result is a form of low-pass filtering.

In order to test these ideas, consider the way we could obtain an actual test signal of the type that was simply sketched in Fig. 7. We could use the inverse CTFT on a rectangular spectrum. For example, taking the bandwidth as being ± 1 Hz ($\Omega = \pm 2\pi$), we calculate:

$$x(t) = (1/2\pi) \int_{-2\pi}^{2\pi} e^{j\Omega t}\, d\Omega = 2 \sin(2\pi t)/(2\pi t) \qquad (16)$$

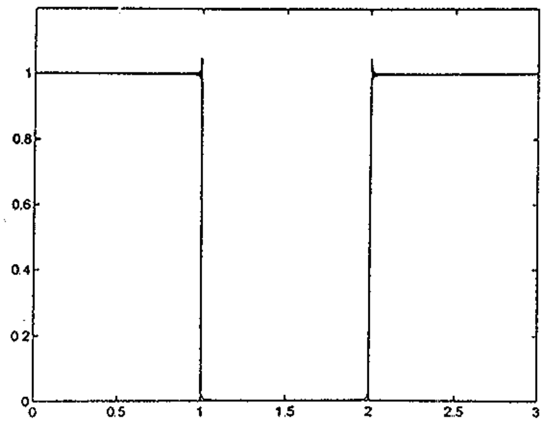



Fig. 8a  Bandwidth=1, $f_S$=3

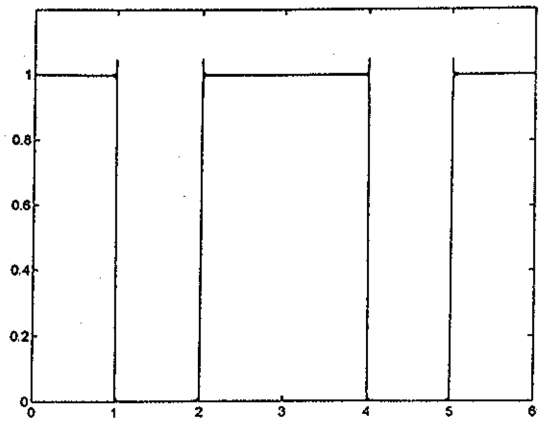



Fig. 8b  Fig. 8a, zero-padded

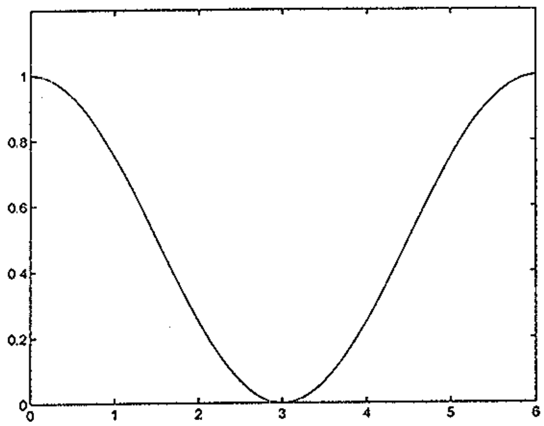



Fig. 8c  Magnitude Response of Linear Interpolator

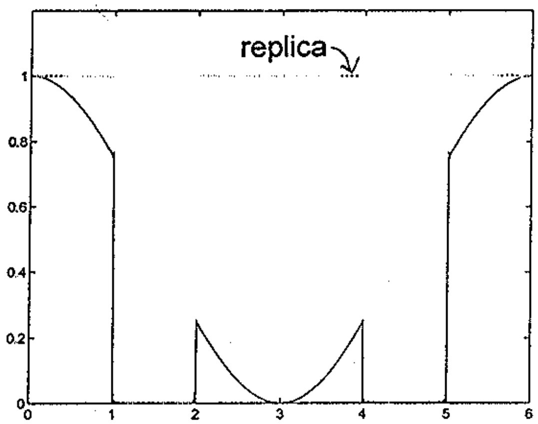



Fig. 8d  Zero-padded with Linear Interpolation

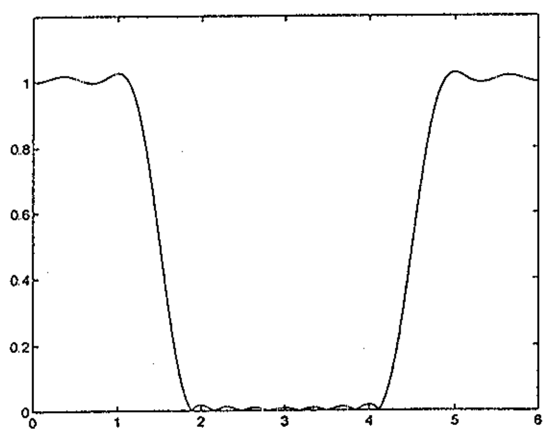



Fig. 8e  A Length 17 Filter

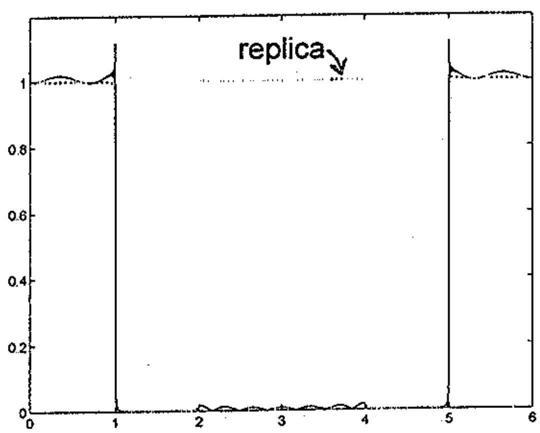



Fig. 8f  Zero-padded Filtered by length 17 Filter

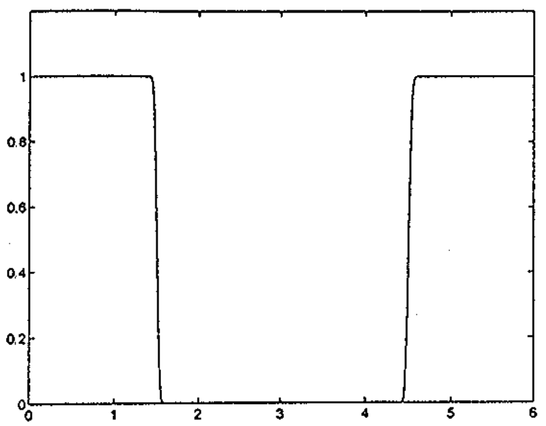



Fig. 8g A Length 201 filter

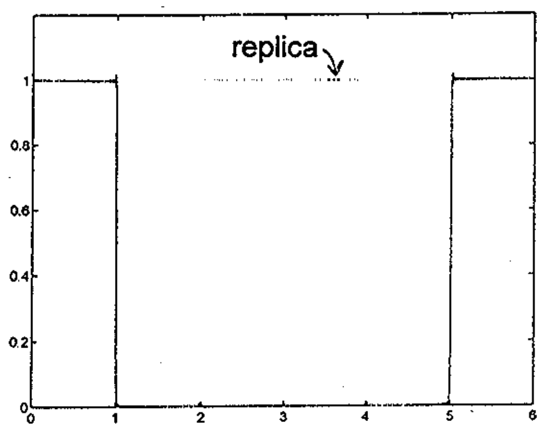



Fig. 8h Zero-padded filtered by Length 201 filter

which we then want to sample at 3 Hz. Thus we evaluate x(t) at intervals of 1/3. Fig. 8a was obtained using 601 samples of equation (16) centered about 0. We then take the magnitude FFT of the 601 time samples, and frequency-calibrate to the range 0 to 3 Hz. [Note the "Gibbs phenomenon" peaking on the edges that are due to truncation.]

The next step is to "zero pad" the samples, placing zeros between the original samples. The magnitude FFT of this length 1202 signal is seen in Fig. 8b, and is seen to correspond nicely to Fig 7. This zero-padded sequence is then convolved with the sequence 1/2, 1, 1/2, with the resulting spectrum shown in Fig 8d. Here we immediately note that the rectangular spectrum has been shaped by the raised cosine of equation (15), Fig. 8c. We see that this linear interpolation has resulted in a digital low-pass filtering, but not a very good filtering. We have rounded off the images we want to keep and only taken a big chunk out of the one we want to remove.

Clearly we need a better digital low-pass filter. One approach (which is interesting and productive at times) is to start with a better time-domain interpolator (such as fitting a higher order polynomial rather than just a straight line to the data). Here however we will just use a standard method (minimized integrated squared error in the frequency domain) and just use the filter, not being concerned with the actual design.

Fig. 8e shows the magnitude response of a particular length 17 low-pass filter. When we convolve the impulse response of this filter with the zero-padded input, we get an output sequence which has a magnitude FFT as shown in Fig. 8f. Note that this is a much better attempt at removing the spectral replica centered about 3. We still clearly see the shape of the filter (Fig. 8e) in the spectrum (Fig. 8f). Finally, Figures 8g and 8h show the corresponding results for a length 201 filter. This final result would seem to be totally satisfactory. In cases where more improvement is needed, we can oversample by larger factors. For example, oversampling by 16 or more in CD players is common.

# 3. VARIATIONS ON ORDINARY SAMPLING

Understanding is almost always improved when we go beyond material we absolutely need to know to more difficult material. In looking at some unusual sampling procedures, we have this idea of obtaining deeper understanding in mind, but also these procedures are at times real occurrences in practice. Perhaps the most important idea is that of a "conservation of information." Do we have enough numbers (per second) to recover our **signals [1]?**

## 3a. GATED SAMPLING

### 3a-1 Gated Sampling and Sample-And-Hold Compared

In this section, we will look at the procedure of "gated sampling" which occurs in analog multiplexing. This we will compare with sample-and-hold (S&H) which we looked at in Section 2d.

In gated sampling, we do not take point samples, but instead the "samples" are small segments of the actual analog waveform (Fig 9f). These are the result of multiplying the original continuous-time waveform (Fig. 9a) by the periodic gate (Fig. 9e). The usual motivation here is obtained by recognizing that $\tau$ can be substantially smaller than T. For example, if $\tau = T/5$, we could try to simultaneously sample five different analog waveforms, each offset for no overlap, and all five could be summed and transmitted down a single channel. This is classical analog multiplexing. At the far end of the channel, the five signals could be separated by a de-multiplex procedure, and the full waveforms reconstructed (we need to show that this is possible). Note the simplicity of the "sampling" here: we do not really need analog multipliers, but just switches controlled by the gates. Nor is A/D conversion involved. We do need to be sure we can reconstruct the signal in the wide gaps.

**An interesting comparison here is the similar-looking S&H (Fig. 9d) which we obtain by first point sampling the analog waveform. Unlike the original S&H of Section 2d, here we will assume we are using a shorter hold time, $\tau$<T instead of $\tau$=T. Our motivation for this might well be to alleviate the problem with the roll-off of the original S&H (Fig. 6). The spectrum with the shorter hold time is still shaped by a sinc, but the roll-off is more gradual, as seen in equation (17) In fact this might be used in practice, but the shorter rectangles will mean less total energy into a reconstruction low-pass filter, and the signal-to-noise ratio can suffer.**

The difference between Fig. 9d and Fig. 9f is thus a matter of the shape of the tops of the "pulses," being perfectly flat in Fig. 9d and being curved, in some manner, in Fig. 9f. The case of Fig. 9f clearly involves more information, and we hope to gain some advantage from this excess. While we do expect spectral images from any sampling procedure, note that it is clear that as $\tau \rightarrow T$ these must disappear in our mathematics, because at $\tau$=T the continuous-time waveform comes back unchanged.

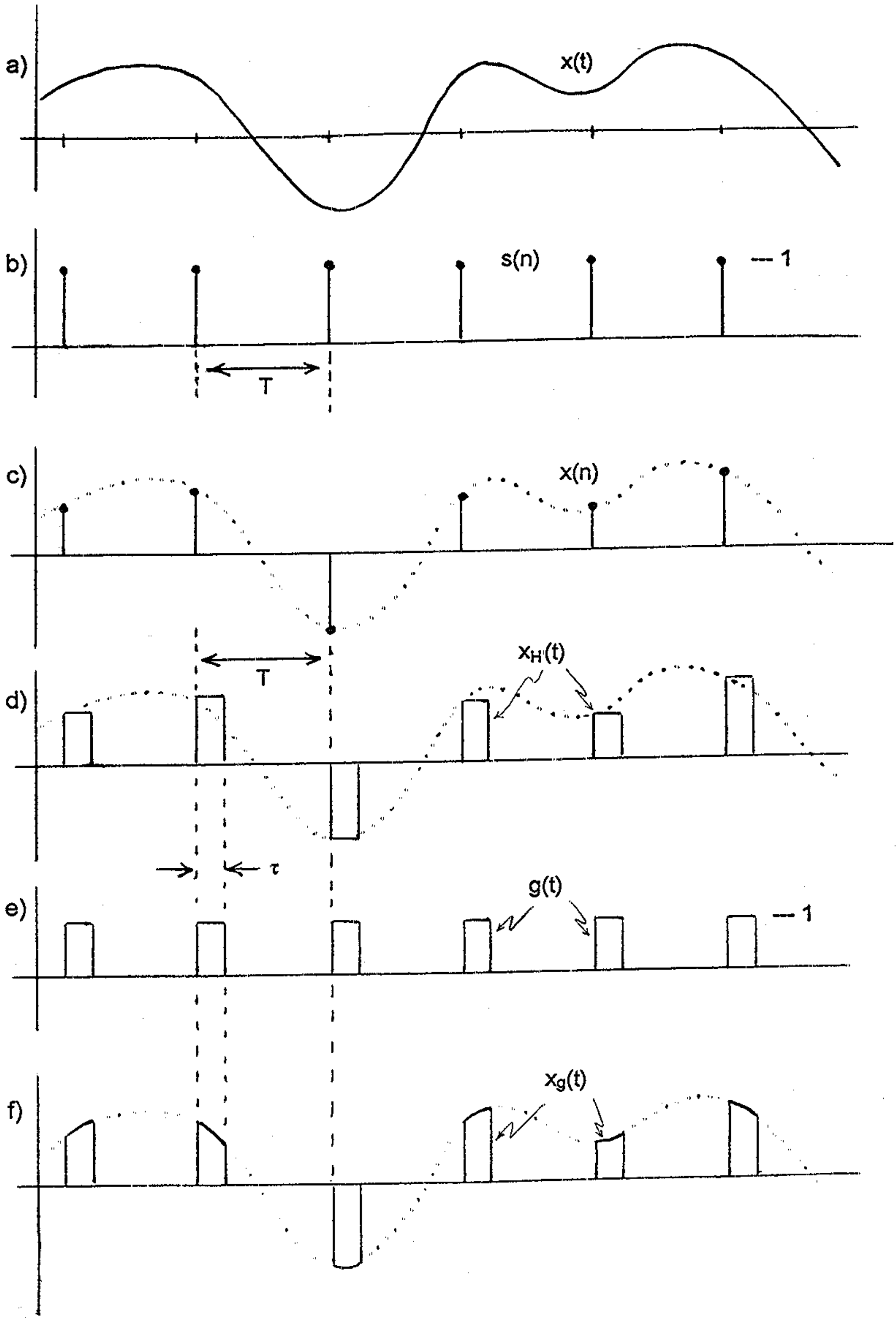



Fig. 9 Sample-and Hold and Gated Sampling Compared

Here it will be useful to "shorten up" the mathematics a bit. Consider r(t) to be a rectangular pulse of width $\tau$ centered at 0. The CTFT of r(t) is $R(\Omega)$ given by:

$$R(\Omega) = \tau \sin(\Omega\tau/2) /(\Omega\tau/2) \tag{17}$$

which is a variation on equation (13a). Borrowing the "shah" symbol [2],

$$Ш(x) = \sum_m \delta(x-m) \tag{18}$$

is a delta-train. Thus we can multiply x(t) by Ш(t/T) and have a correct representation of x(n), <u>as long as</u> we intend to integrate over the x(t)Ш(t/T) terms, which will happen when we use convolution. Similarly, we can represent the gate g(t) as Ш(t/T)*r(t) where the "*" denotes convolution. In addition, similar representations using the shah are found in the frequency domain. Note in particular the CTFT pair:

$$Ш(t/T) \text{ <----> } T\, Ш(fT) \tag{19}$$

so as expected, a spacing of T in time corresponds to a spacing of 1/T in frequency.

We first note that:

$$x_H(t) = [\, x(t)\, Ш(t/T)\, ]*r(t) \tag{20}$$

which is ordinary point sampling followed by convolution with the hold rectangle, while:

$$x_g(t) = [\, r(t)*Ш(t/T)\, ]\, x(t) \tag{21}$$

where we first generate g(t) by convolving a shah and the rectangle, and then multiply by x(t). In the frequency domain, multiplication and convolution change places, so that:

$$X_H(\Omega) = T\, [\, X(\Omega)*Ш(\Omega T/2\pi)]\, R(\Omega) \tag{22}$$

and

$$X_g(\Omega) = T\, [R(\Omega)\, Ш(\Omega T/2\pi)] * X(\Omega) \tag{23}$$

This "fun" set of relationships tells us that when we use the S&H, the original spectrum $X(\Omega)$ is first replicated by convolution and then shaped by $R(\Omega)$, the sinc roll-off. In the gated sampling case, however, Ш($\Omega T/2\pi$) is multiplied by $R(\Omega)$ first, and then the spectrum $X(\Omega)$ is replicated by convolution with a delta-train that rolls-off. Thus in the case of gated sampling, the spectral replicas are not distorted, having the exact same shape; only their amplitudes are modified.

Note that as $\tau \to 0$, in either the gated case or the S&H, $R(\Omega) \to 1$ and both cases revert to point sampling. As $\tau \to T$, in the gated sampling case, $R(\Omega)$ is 0 except at $\Omega=0$, and the continuous-time case (no replicas) returns.

## 3a-2 The Extra Information in Gated Sampling

Above we found that the spectral images obtained with gated sampling are perfect copies of the original continuous-time spectrum except for overall multiplication factors that are known - calculated from a sinc function, equation (17). This was in contrast to the S&H case where the set of images (or any sum of overlapping images) was distorted by the sinc function. Yet the S&H roll-off is relatively easy to correct (most DSP evaluation cards that plug into computers offer this feature). So gated sampling and S&H are mainly compared here for their symmetrical formulas, but are otherwise quite different and are unlikely competitors for any actual application.

As suggested, there is one immense difference between the two which needs to be further explored: the gated sampling case has "extra samples" (extra information) in reserve. Essentially we refer to the fact that point samples (or the held levels with S&H) have no information other then the values of the samples themselves. The tops of gated samples, in contrast, are of course not flat in general, being actual portions of the original continuous-time waveform, and this is extra information - theoretically an unlimited reservoir of extra information. This will offer us the opportunity to discard gated samples (equivalent to downsampling) and still recover the original signal, even if the lower sampling rate (reciprocal of interval between kept gated samples) is insufficient for the bandwidth.

To begin to understand this we need to recognize that the "sampling theorem" is much more general than the usual form in which we find it first presented (as above). Indeed we should think more in terms of "conservation of information," sometimes referred to as a "dimensional theorem" which states the number of pieces (dimension) of information needed to reasonably specify a waveform: $N \cong 2BT_0$, where B is the bandwidth in Hz and $T_0$ is the length of the signal. This effectively includes the usual sampling theorem if we rewrite it as a rate: $N/T_0 > 2B$. The difference is thinking in terms of a finite-length signal, some notion of usable approximations instead of exact (impractical) reconstruction, and no pre-specified notion of the N pieces of information (in particular, they need not be equally spaced samples). This is used for "back of the envelope" calculations of feasibility.

Perhaps the notion of orthogonal expansions is the most familiar idea relating to what we are suggesting here. It is perfectly possible to think of the usual bandlimited sampling/reconstruction ideas as a sinc expansion with the samples themselves being the coefficients of the expansion [equation (12b)]. We shall see in Sections 3b and 3c how alternative basic functions for the expansion appear relative to variations on conventional sampling.

But, back to the question of the extra information that we believe must be present in gated sampling. There are two ways in which we can see that gates sampling can undergo downsampling without loss of information. The first method is to argue that if we have the gated samples, we can always take point samples within the gated samples, as in Fig. 9, where we show two such samples per gate. Making the usual assumption that the bandwidth in Hz is less that 1/2T, it is clear from the studies above that either the gated samples themselves, sample set A, or sample set B is sufficient to recover the original signal. What is not so well known (as will be seen in Section 3b) is that the signal can also be recovered from sample set C. That is, we can keep two samples within one gate and cast out the two samples in the next gate, and so on, effectively throwing away (downsampling) every other gated sample. It is the average sampling rate that matters. It is further clear that we can, at least in theory, continue this idea indefinitely, taking three samples per gate and casting out two of three gates, and so on.

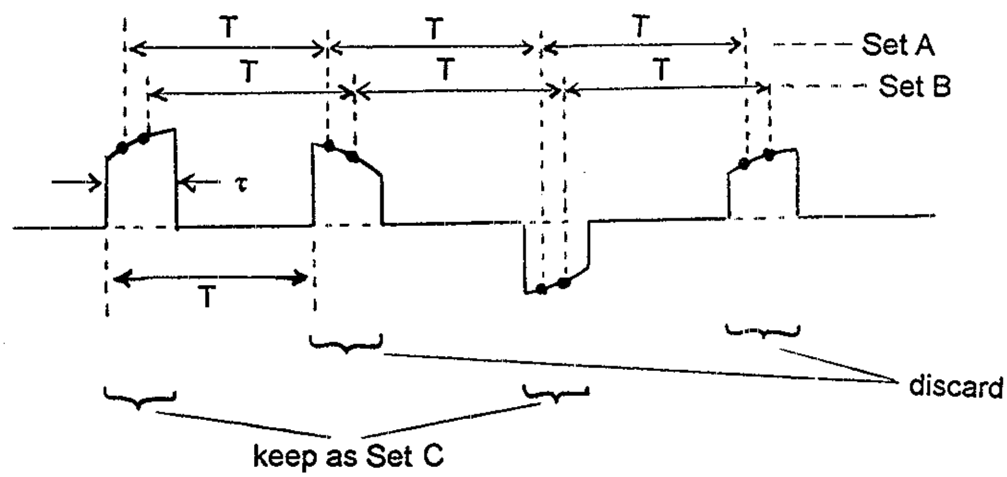



Fig. 9 Gated Sampling as a Reservoir for Points Samples

The second way in which we can see that we can downsample the gated sampling and still recover the original signal is to observe that the original spectrum can be recovered even after overlap. Suppose that we have an original spectrum as shown in Fig. 10a, and that this spectrum is "full-bandwidth" reaching all the way to $f_S/2$. We thus have spectral copies about $f_S$, $2f_S$, and so on, each maintaining the shape of the original, but having different amplitudes [using equation (17)]:

$$A_k = \tau \sin(\pi k\tau/T) / (\pi k\tau/T) \tag{24}$$

For example, if $\tau$=T/4 so that the gate is on 1/4 of the time, then the amplitudes are:

| $k$ | $A_k$ |
|---|---|
| 0 | $\tau$ |
| 1 | $0.9003\tau$ |
| 2 | $0.6366\tau$ |
| 3 | $0.3001\tau$ |
| 4 | 0 |
| 5 | $0.1801\tau$ etc. |

Next suppose that we discard every other gate. This is the same as sampling at $f_S/2$, or having a sampling interval of 2T. We will then get spectral replicas that overlap, as in Fig. 10b. Here $\tau$ is still T/4 in our example, so the amplitudes are different, but still determined in the same manner from equation (17):

| $k$ | $B_k$ |
|---|---|
| 0 | $\tau$ |
| 1 | $0.9745\tau$ |
| 2 | $0.9003\tau$ |
| 3 | $0.7842\tau$ |
| 4 | $0.6366\tau$ |
| 5 | $0.4705\tau$ |
| 6 | $0.3001\tau$ etc. |

The point is that we have complete overlap at this point, and the spectrum of interest from 0 to $f_S/2$ is corrupted, being the sum of the original and the replica centered at $f_S/2$. Can we recover the original spectrum X(f) from the sum W(f)? [Here we are assuming that original signals are real. If they are complex (not a common case, but sometimes encountered) than we would need twice the information (real and imaginary parts at every sample), and the argument here would need to be modified.]

Note that we do know the constants $B_0$ and $B_1$, and we have the sum W(f) for all f between f=0 and $f=f_S/2$. For a particular frequency f in this interval, we have (Fig. 10b):

$$B_0\,X(f) + B_1\,X(f_S/2-f) = W(f) \tag{25a}$$

and it is also seen that:

$$B_0\,X(f_S/2-f) + B_1\,X(f) = W(f_S/2-f) \tag{25b}$$

We now have two linear equations in two unknowns which we can solve for X(f) and $X(f_S/2-f)$. Solving for X(f) we have:

$$X(f) = \frac{B_0\,W(f) - B_1\,W(f_S/2 - f)}{B_0^2 - B_1^2} \tag{26}$$

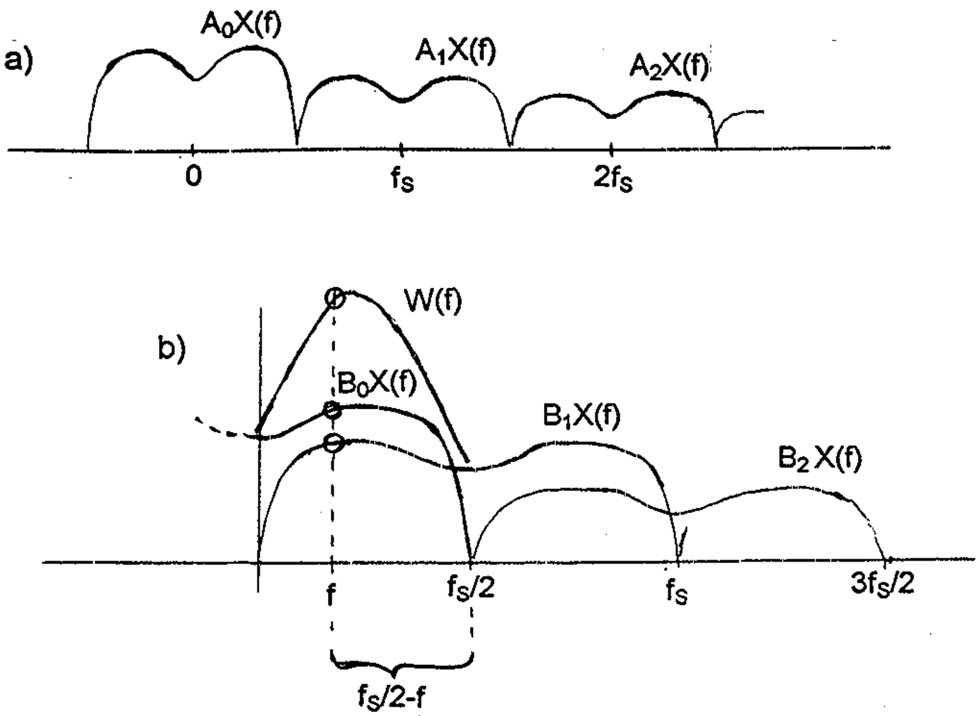



Fig. 10 In (a), we see spectral replicas with the exact same shape, which do have different amplitudes $A_k$. In (b) we have downsampled the gated samples so the spectra overlap. These are still of exactly the same shapes, and the amplitudes $B_k$ are known.

This tells us that we can recover X(f) from W(f) since we know $B_0$ and $B_1$ and since $B_0 \neq B_1$. If $B_0=B_1$, we would have the spectrum that corresponds to point sampling. In this case, the equations are ambiguous and can not be solved.

It is easy to write the equations above and discuss the solution. In fact in general it is much easier to argue that it is possible to recover a signal from some variation on normal sampling than it is to actually do it. What we would actually need to do here would be to separately filter out the spectral portions from f=0 to $f=f_s/4$, and from $f=f_s/4$ to $f=f_s/2$, shift these by modulation methods, and then add everything back together. It is not unusual for the "overhead" to wipe out any advantage. Of course, the problem gets worse if we try downsampling by numbers greater than 2. However, we see that gated sampling can be "downsampled" in a lossless manner indefinitely!

## 3b. SAMPLING IN PAIRS / AN AVERAGE SAMPLING RATE

Above in Section 3a-2 we discussed gated sampling and mentioned that one way to understand downsampling of gated samples was to consider taking samples in close pairs within the gate (promising to give the details later). Here we want to look at this procedure of taking samples in pairs [2]. We can start with the premise that we are taking samples at intervals T where the bandwidth of the signals is 1/2T - ordinary sampling obeying the ordinary sampling theorem. Now suppose that we continue to take a first set of samples, Set A, at even multiples of T (0, 2T, 4T, 6T, etc.) but that the second set of sample at odd multiples of T (T, 3T, 5T, ... etc.) are not taken. Instead, a different second set of samples, Set B, is taken, at times displaced from the first sampling times by a time interval $\alpha$ ($\alpha$, 2T+$\alpha$, 4T+$\alpha$, etc.). We now have two sets of samples, Set A and Set B. Neither set can be used alone to recover the original signal. This is clear since either set, taken at a sampling rate of 1/2T, is insufficient to support the 1/2T bandwidth Neither would we have any success (as long as $\alpha \neq T$) of low-passing the sets individually and summing the results. The one thing we would perhaps cling to is the fact that there are enough total samples, on average. There may well still be enough information It is just that the recovery procedure is expected to be more complicated.

A more complete description of the recovery procedure is offered in Bracewell [2]. Here we will note that the spectra corresponding to the two sets are overlapped, and we know tnat portions of the two spectra, corresponding to a delay in time, are related by an exponential factor $e^{-j2\pi\alpha}$ in the frequency domain (standard CTFT properties). In much the way we were able to solve for the original spectrum in the case of gated sampling (Section 2a-2), Bracewell solves the paired sampling case. Further, he then jumps back to the time domain so that we can reconstruct by convolution, in the manner used for sinc interpolation [equation (12a)] The difference is that the interpolating functions are not the usual sincs; rather we get two interpolating functions. The first is:

$$a(t) = \mathrm{sinc}(2t) - \beta\, t\, \mathrm{sinc}^2(t) \qquad (27a)$$

and

$$b(t) = a(-t) \qquad (27b)$$

where the constant $\beta$ is:

$$\beta = \pi/\tan(\alpha\pi) \qquad (28)$$

and where the sinc function is here defined as:

$$\mathrm{sinc}(x) = \sin(\pi x)/(\pi x) \qquad (29)$$

The encouraging thing about these result is that the interpolating functions a(t) and b(t) contain terms in sinc(2t) and $\mathrm{sinc}^2(t)$, both of which are capable of spawning waveforms that "wiggle" twice as fast as sinc(t) itself. This is important in as much as interpolating with sinc(t), with the undersampled original Sets A or B, produces an aliased (lower frequency) waveform, and we are certainly going to need something that generates some faster components.

In actual use, the original waveform is obtained by convolving the samples sets with their corresponding interpolating function:

$$x(t) = a(t)*x_A(n) + b(t)*x_B(n) \quad (30a)$$

$$= a(t)*[\, x(t)\, Ш(t/T)\, ] + b(t)*[\, x(t)\, Ш(t/T-\alpha)\, ] \quad (30b)$$

where $x_A(t)$ and $x_B(t)$ are Set A and Set B respectively. The interpolating functions a(t) and b(t) are plotted in Fig. 11. Note that in the actual interpolation (convolution) of equation (30b), the interpolating function b(t) is appropriately offset by $\alpha$.

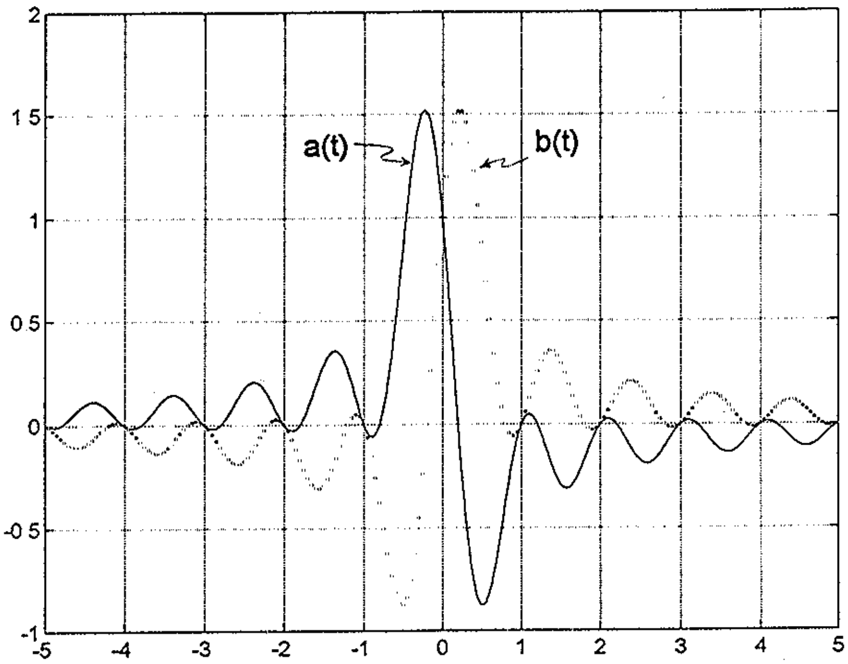



Fig. 11 The interpolating functions for paired sampling ($\alpha$=0.2T)

Fig. 12 shows an actual example. Here we have a sinusoidal waveform of frequency 2/3 sampled by Set A (*) and by Set B (o) at a frequency 1. The interpolation using equation (30b) is seen to still be a good match to the original waveform. [The ends show more discrepancy than the middle. and these end effects are due to truncation error and are similar to those we saw in Fig. 4b.] For comparison, we can interpolate only set A using a sinc(t) interpolation function (ideal low-pass with cutoff 1/2), and this is seen to be the lower frequency 1-2/3= 1/3.

## 3C SAMPLING THE SIGNAL AND ITS DERIVATIVE

Above we have spent considerable time on sampling the signal itself at regular intervals. Just above in Section 3B we have seen that samples at alternating, irregular intervals are possible. Here we will look at the simultaneous sampling of a signal and its (first) derivative.

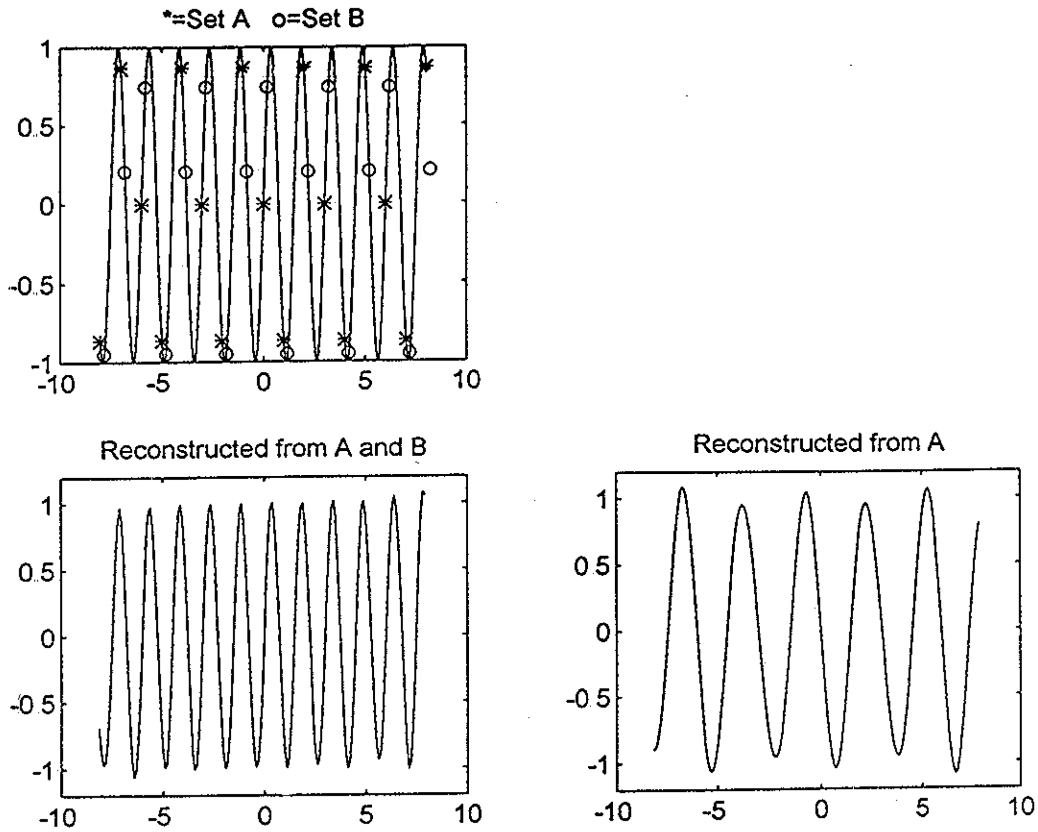



Fig. 12 Here a sineusoidal waveform of frequency 2/3 is sampled at a sampling frequency of 1, at the integers (o) and separated 0.2 from the integers (*). Using both sets of samples, the sinusoidal waveform is recovered at frequency 2/3. Using only one set of the samples, the aliased sinusoidal waveform of frequency 1/3 is found.

A few opening comments are perhaps useful. First, if we attempt this, we could use a simple analog circuit to produce the derivative and then sample this analog derivative. Secondly, thinking in terms of the rate of information, an average sampling rate of sorts, the simultaneous sampling of the signal and its derivative give us twice the information rate, and we expect to be able to support twice the bandwidth. Finally, we might well expect this to work since it would seem to be a limiting case of paired sampling as $\alpha \rightarrow 0$.

Here, as in the case of paired sampling, we are not overly interested in recovering signals from extra information by a complicated process if recovery by a simple process without the extra information is perfectly viable. Thus in sampling both the signal and its derivative we are interested in sampling both at intervals T while working with bandwidths that extend into the region of frequency from 1/2T to 1/T. That is, in sampling two things rather than just one we hope to be able to double the bandwidth or else use only half the sampling rate. (To better appreciate this, suppose that we are sampling in a spatial dimension along a line on the surface of the earth - "tramping through the wilderness." We might well find it more convenient to take multiple samples at an accessable location rather than have to reach a difficult location.)

The setup for this case, called ordinate-and-slope sampling by Bracewell [2] is not unlike that for the paired sample case. Here CTFT properties relating to differentiation as seen in the frequency domain (a multiplication by frequency itself) can be employed. Again, Bracewell solves for the original spectrum, and transforms the results into interpolation functions and an interpolating procedure (via convolution) back in the time domain.

$$x(t) = a(t)*x(n) + b(t)*x'(n) \tag{31}$$

where x'(n) are the samples of the derivative. There the interpolating functions are:

$$a(t) = \mathrm{sinc}^2(t) \tag{32a}$$

and

$$b(t) = t\ \mathrm{sinc}^2(t) \tag{32b}$$

and we are again pleased to see functions capable of producing "wiggles" faster than sinc(t) itself. Putting equation (31) into a bit more concrete form, we have:

$$x(t) = a(t)*[\ x(t)\ Ш(t/T\ ] + b(t)*[\ x'(t)\ Ш(t/T)\ ] \tag{33a}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\mathrm{sinc}^2(t-n) + \sum_{n=-\infty}^{\infty} x'(n)(t-n)\mathrm{sinc}^2(t-n) \tag{33b}$$

In the case of the paired samples, our actual demonstration was in terms of a fixed sample rate (of 1) followed by our choice of a frequency (2/3) that would have caused aliasing with ordinary frequencies. In this ordinate/slope example, we will choose a different sort of display, mainly for variety. Here we will hold the frequency of our signal fixed (at 0.1), and then we will systematically lower the sampling rate.

For this test, we choose a sinusoidal waveform, so we will also show the cosine to indicate the form of the derivative. In addition here we have tapered the ends of the signal so as to reduce end effects due to truncation. For each of the three sampling rates chosen, we will show the signal as we attempt to recover it from the samples alone, and as we recover it from both the samples and the samples of the derivative.

In Fig. 13a, we choose a sampling rate of 1/3=0.3333..., greater than twice the frequency of the sinewave (twice 0.1 = 0.2). We see successful and comparable recovery in both cases. In Fig 13b, the sampling rate is 0.2, exactly twice the frequency of the sinewave. For this example, note that the samples themselves are all zero -- they just happen to be zero, but they are perfectly valid samples.

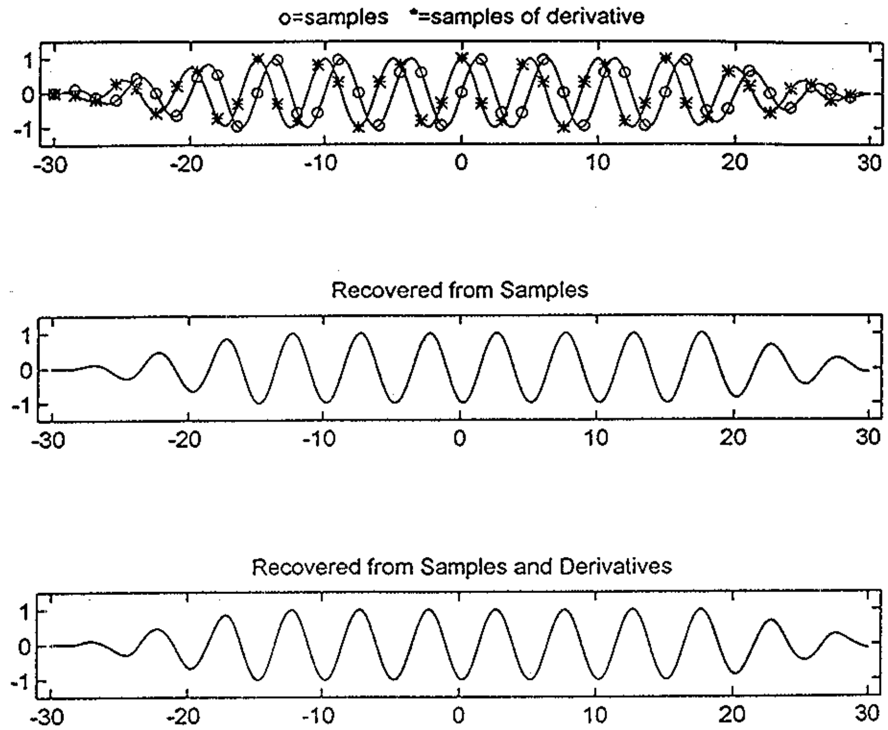



Fig. 13a Sampling a sinusoidal of frequency 0.1 and its derivative at 1/3=0.3333.... allows recovery from the samples or from the samples and the samples of the derivatives

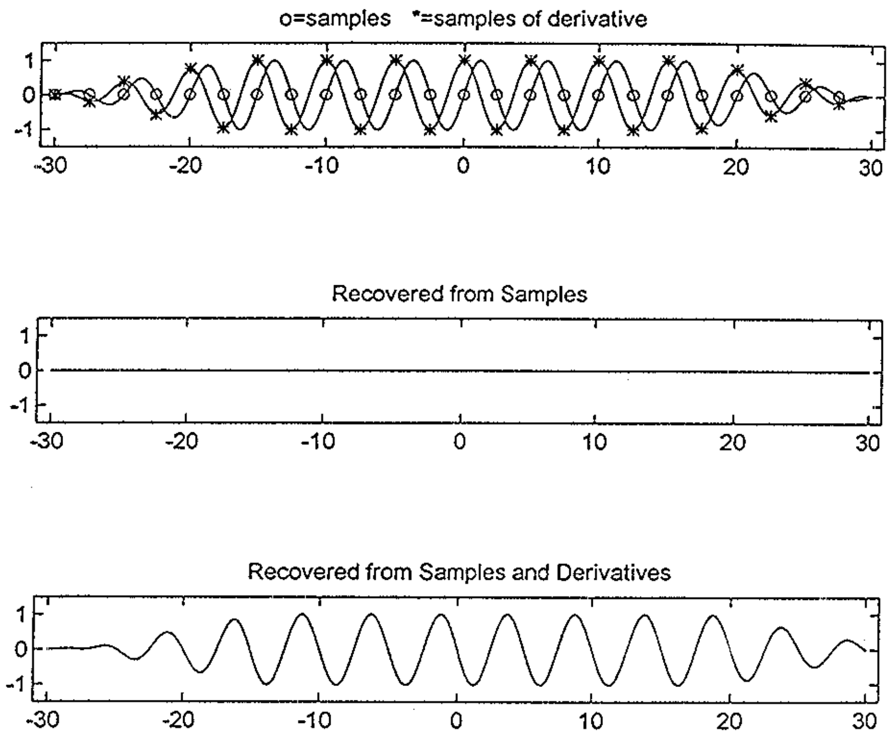



Fig. 13b Sampling a sinusoidal of frequency 0.1 and its derivative at 1/5=0.2 (critical case allows recovery using both the samples and the samples of the derivatives

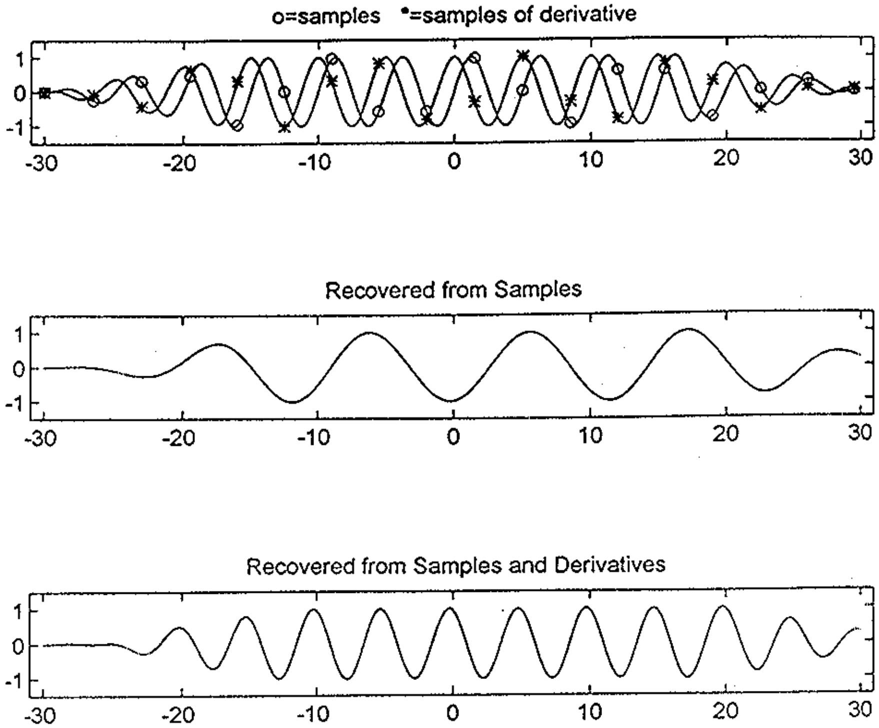



Fig. 13c Sampling a sinusoidal of frequency 0.1 and its derivative at 1/7 = 0.14286 allows recovery using both the samples and the samples of the derivatives. Attempting to recover using only the samples aliases to 0.04128 = 3/70 with reconstruction cutoff at 1/14.

Finally for Fig. 13c, the sampling rate is **1/7 = 0.14286,** less than twice the frequency of the sinewave. Here the attempted recovery from the samples themselves gives an aliased result, while recovery from the samples and the derivative continues to return the correct frequency.

# 4. RESAMPLING, COMPRESSION, DOWNSAMPLING, DECIMATION, EXPANSION, UPSAMPLING, AND INTERPOLATION

## 4a. INTRODUCTION

The problems discussed above have for the most part been concerned with sampling in the sense that a particular continuous-time signal existed and served as the source of samples, and also as a goal for reconstruction from these samples. But there are also situations where we have already achieved sampling, and we then want to make adjustments to the resulting sequence of numbers. The sort of things we may want to do typically involve the removal of some samples (the related processes of "resampling," "downsampling," and "decimation") or to add samples (the related processes of adding zeros to the sequence, "upsampling," and then filling in or "interpolating" the zero values to more useful values).

One view which we can often make use of is to observe that in manipulating one sequence to another, we can usually employ a "what if" approach. We start with one sequence, obtained from a sampling process which we understand, and do something to this sequence, creating a new sequence. What if this second sequence had been obtained directly by use of the original sampling process? Typical of this approach is the observation that if we take samples, and then throw every other sample away, somehow we can understand this usefully in terms of sampling the original signal at half the original sampling rate. The history does not matter. This view often offers us a "reality check" of considerable value.

Possibly nothing is more injurious to relatively easy understanding of the material here than the problems associated with terminology, and with frequency conventions. Associated with the idea of discarding some samples from an original sequence are the terms decimation, downsampling, and compression (and likely others). Associated with the idea of adding samples to an original sequence are the terms interpolation, upsampling, and expansion (and likely others). Different textbooks may use these terms in different ways. One must be extremely careful to not infer too much by the use of a particular term, but to instead rely on the actual description of what is done.

[We can not hope to suggest a resolution to these terminology problems, or even offer a full catalog, except to perhaps suggest that priority might well be given to the terms expansion and compression, and to insist that the term interpolation should never be used for the case where zeros are inserted in a sequence, but not then simultaneously replaced with more suitable non-zero values. One saving thing seems to be the relatively consistent use of a symbol or notation, usually seen in block diagrams, that employs a number and an associated arrow pointing up or down, usually enclosed in a box. But even this occasionally deteriorates. Buyer beware!]

And this issue of terminology is only the first source of confusion. As mentioned, there are also problems with frequency conventions. Prior to the widespread exposure to the extremely useful notions found in the "multi-rate" DSP art, it was not unreasonable to consider a "normalized" sampling rate. (This normalization still is exceedingly useful in digital filter design, for example.) Typically this normalization was chosen as one sample per second (1 Hz) corresponding to $2\pi$ radians per second. Subsequently the actual dimensions (inverse second) were forgotten so that the $2\pi$ sampling rate became dimensionless. With the idea of indexing equally spaced samples as a sequence, defined on the integers, it made sense to have the sampling rate always be a dimensionless 1 (or $2\pi$). But with the advent of multi-rate DSP, we had the contradictory situation where the sampling rate was always the same, and where we had more than one sampling rate. This leads to the need to constantly renormalize at each step. In consequence some counterintuitive scalings of the frequency axis have appeared.

Here we will insist that it is necessary to understand things in terms of both physical frequencies and normalized frequencies. While we believe that physical frequencies are a much better choice for engineers, it is unquestionably also true that the normalized (purely mathematical) presentation is so prevalent that it too must be understood.

In our presentation, we will first define some terms as best we can. We will then develop the mathematical tools of resampling, compression, and expansion. Next we will compare and contrast the physical and mathematical cases, pointing out that they lead to equivalent results. Finally, some additional downsampling examples will be presented.

## 4b. SOME TERMINOLOGY AS WE SEE IT

### 4b-1 Reducing the Number of Non-zero Samples

RESAMPLING: The input sequence is multiplied by a resampling sequence of the same length as the sequence (or nearly so). For example, the resampling sequence [1 0 0 1 0 0 1 0 ......] sets two of three samples to zero. The original sequence and the resampled sequence are the same length For example, if the original sequence is [a b c d e f g h i j k l] the resampled sequence would be [a 0 0 d 0 0 g 0 0 j 0 0].

COMPRESSION: We keep only every $N^{th}$ sample of the input sequence. The output sequence is only 1/N as long (or nearly so).

DOWNSAMPLING: This can be simply the same thing as compression It can also refer to the actual change of physical sampling rate. For example, the full and complete conversion of sampling rate from 50 kHz to 40 kHz.

DECIMATION: Again, this can be the same thing as compression. It can also refer to a physical situation where the sampling rate is reduced. In such a case, it is usually the case that a pre-decimation filter reduces the bandwidth so that no aliasing results from the loss of samples.

### 4b-2 Adding to the Number of Samples

EXPANSION: Zeros are inserted between all samples of the original sequence. Expansion by N adds N-1 zeros between existing samples and the resulting sequence is N times as long. For example, expanding [a b c] by 3 gives [a 0 0 b 0 0 c 0 0].

UPSAMPLING: Usually, the same as expansion. Used in the case of physical frequencies, it may refer to the inverse of downsampling, a full and complete conversion from 40 kHz to 50 kHz for example.

INTERPOLATION: Not expansion. It is expansion followed by a replacement of the inserted zeros with replacement samples, perhaps through a time-domain interpolation procedure, or equivalently, through a frequency-domain low-pass filtering to remove sampling replicas.

## 4c. MATHEMATICAL DERIVATIONS

### 4c-1 A Mathematical Form for a Resampling Function

The idea of a resampling function $s_N(n)$ can be presented as:

$$s_N(n) = \ldots\ \underbrace{0\,0\,1\,0\,0\,0}_{\text{length } N}\ \ldots\ 0\,0\,1\,0\,0\,0\ \ldots\ 0\,0\,1\,0\,0\,0\ \ldots \qquad (34a)$$

One period of $s_N(n)$ is of course just:

$$s_N(n) = [\,1\ 0\ 0\ 0\ \ldots\ 0\ 0\,] \qquad n=0,1,\ldots(N-1) \qquad (34b)$$

where we use the same notation for one period since we are about to take the DFT anyway. The DFT of $s_N(n)$ is $S_N(k)$ which is 1 for all k as is easily seen:

$$S_N(k) = \sum_{n=0}^{N-1} s_N(n)\, e^{-j(2\pi/N)nk} \equiv 1 \qquad (35)$$

Then we can take the inverse DFT to get $s_N(n)$ back:

$$s_N(n) = (1/N) \sum_{k=0}^{N-1} S_N(k)\, e^{j(2\pi/N)nk}$$

$$= (1/N) \sum_{k=0}^{N-1} e^{j(2\pi/N)nk} \qquad (36)$$

The advantage of equation (36) over equation (34a) is that it is in a usable mathematical form.

### 4c-2 Resampling

Given a time signal x(n), its resampled form $y_S(n)$ is:

$$y_S(n) = x(n)\, s_N(n) \qquad (37)$$

By definition, the DTFT's of x(n) and $y_S(n)$ are:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jn\omega} \qquad (38)$$

$$Y_S(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y_S(n)\, e^{-jn\omega} \tag{39}$$

We can substitute equation (37) and equation (36) into equation (39):

$$Y_S(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\,(1/N) \sum_{k=0}^{N-1} e^{j(2\pi/N)nk}\, e^{-jn\omega} \tag{40}$$

which can be rearranged as:

$$Y_S(e^{j\omega}) = (1/N) \sum_{k=0}^{N-1} \sum_{n=-\infty}^{\infty} x(n)\, e^{-j[-(2\pi/N)k+\omega]n} \tag{41}$$

which we then recognize as:

$$Y_S(e^{j\omega}) = (1/N) \sum_{k=0}^{N-1} X(e^{j[\omega-(2\pi/N)k]}) \tag{42}$$

This equation simply indicates that the spectrum of the resampled signal is the spectrum of the original, scaled by 1/N, plus N-1 scaled replicas inserted in-between. This is entirely consistent with the notion of a sampling rate lowered by N.

## 4c-3 Compression (Down-Box)

In compression, a sequence $y_C(n)$ is obtained from $x(n)$ by the following rule:

$$y_C(n) = x(Nn) \tag{43}$$

This means that every $N^{th}$ sample is kept while all other samples of $x(n)$ are discarded. To avoid certain pitfalls, it is most useful here to recognize that $y_C(n)$ is a compression of the resampled sequence $y_S(n)$. That is:

$$y_C(n) = y_S(Nn) \tag{44}$$

Thus we write:

$$Y_C(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y_C(n)\, e^{-jn\omega} \tag{45}$$

$$= \sum_{n=-\infty}^{\infty} y_S(Nn)\, e^{-jn\omega} \tag{46}$$

If we substitute m=Nn (n=m/N) we have:

$$Y_C(e^{j\omega}) = \sum_{m=-\infty}^{\infty} y_S(m)\, e^{-jm\omega/N} = Y_S(e^{j\omega/N}) \qquad (47)$$

Now using equation (42):

$$Y_C(e^{j\omega}) = (1/N) \sum_{k=0}^{N-1} X(e^{j[\omega/N - (2\pi/N)k]}) \qquad (48)$$

This is correct, but requires some care evaluating the argument. For k=0, we find that the spectrum is expanded by N. Accordingly, normal sampling replicas at intervals of $2\pi$ are spaced at $2N\pi$. At the same time, the terms for k=1 to N-1 must be added, and these are displaced at intervals of $2\pi$ to fill in expanded spectra at all multiples of $2\pi$. This makes the original spectrum appear merely to expand by a factor of N from 0 and from $2\pi$, toward $\pi$, although the actual mechanism is more involved.

## 4c-4 Expansion (Up-Box)

Happily, the third and last case, expansion, is the easiest case to evaluate. In expansion, all samples are kept, but pushed apart so that N-1 zeros are placed between the original samples. That is:

$$\begin{aligned} y_E(n) &= x(n/N) \qquad && \text{n an integer multiple of N} \\ &= 0 && \text{else} \end{aligned} \qquad (49)$$

$$Y_E(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y_E(n)\, e^{-jn\omega} = \sum_{n=-\infty}^{\infty} x(n/N)\, e^{-jn\omega} \qquad (50)$$

Substituting m = n/N

$$Y_E(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-jNn\omega}$$

$$= X(e^{jN\omega}) \qquad (51)$$

This is a compression of the frequency axis by a factor of N. This compression not only makes the apparent bandwidth smaller by a factor N, but (N-1) spectral repeats, normally at intervals $2\pi$, now appear inside the region 0 to $2\pi$, spaced at $2\pi/N$, and appear similar to sampling replicas. Because there are no discarded samples here, the apparent appearance of sampling replicas may be confusing, and consequently, the true mechanism of their origin should be kept in mind.

## 4d COMPARING THE PHYSICAL AND NORMALIZED VIEWS

Two drawings, which we shall see will lead to essentially equivalent analysis and design conclusions, are seen in Fig. 14a (Normalized Viewpoint) and Fig. 14b (Physical Viewpoint). The normalized viewpoint relates to the mathematics above, while the physical viewpoint perhaps relates better to engineering intuitions and practical implementations. The only real difference will be seen to be that we require the time samples to be defined only on integers in the normalized view, while in the physical view, sample positions are defined in terms of actual time intervals, and in terms of multiples and fractions of these intervals.

The most interesting structures in these diagrams are included in an "Interpolation Path" that runs from bottom to top. We will however begin with a quick discussion of the downsampling path on the left of each diagram. We need to note three further things about these diagrams. We have included a series of letters inside parenthesis so that we can refer to a particular object or region without excessive description. Also, note that the diagrams are drawn for a factor of 2 difference in sample sets. The extension to other factors should be obvious. The final point relates to the Fourier transforms suggested. Fig. 14a is thought of as the ordinary DTFT, equation (4a) while Fig. 14b refers to the physical DTFT, equation (6a)

### 4d-1 The Downsampling Path

In regard to the downsampling paths, we will find it useful and essentially sufficient to employ the "what if" approach suggested above. We can think of the denser sets of samples (A) and (J) as being changed to half-sized sets (H) and (P) by keeping samples a, c, e, .... only. We can then ask what would happen if it were not the full sets (a,b,c,d,e,....) that were obtained by ordinary sampling, but instead, the half sized sets that were obtained by ordinary sampling. This tells us, for example, that we are not going to be able to throw out half the samples unless either (1) the original bandwidth is no greater than 1/4 the original sampling rate, or (2) we first reduce the bandwidth to no more than that value with a low-pass filter, prior to discarding samples. In the case shown here, where we already have taken samples, this reduction of bandwidth, if necessary, would be a matter of employing a digital filter rather than an analog filter (we would use analog if sampling had not yet occurred). Such a digital filter is called a "pre-decimation" filter and would appear in the path from (A) to (C), and from (J) to (P). These are not indicated in the drawings, so at this point, let's assume that we have determined that we are able to cast out half the samples without any resulting spectral overlap. That is, the bandwidth of the original signal was such that less than half (1/3 actually) of the original frequency region was taken up. This is suggested by the sketched spectra, (B) and (K).

This leaves us to consider if the spectra (I) and (Q) following downsampling are correct, and in fact equivalent. Both (H) and (P) show only the samples a, c, e, .... and the corresponding spectra, (I) and (Q), now show 2/3 rather than 1/3 of the total frequency range being occupied. Again, thinking in terms of what happens with

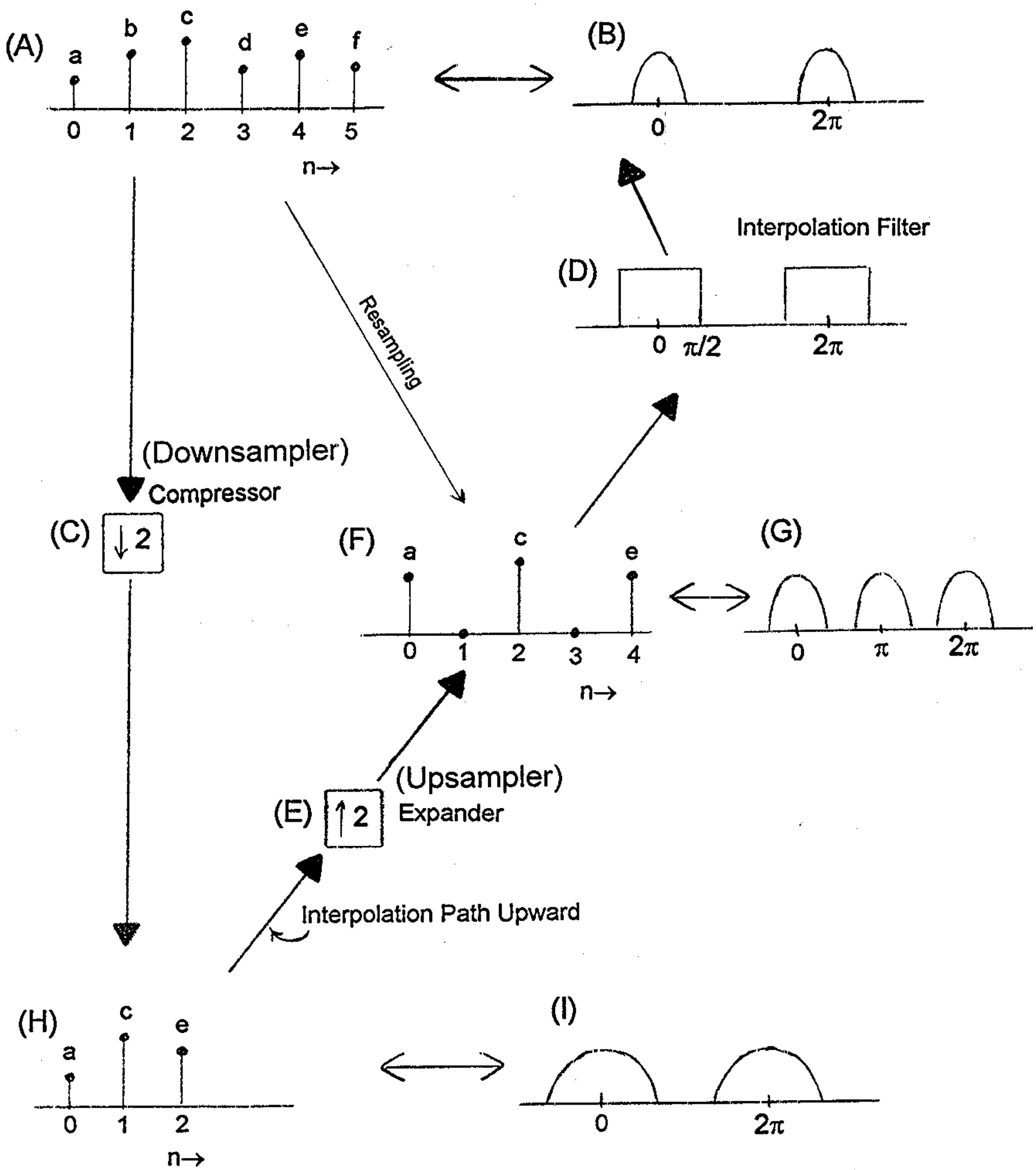



Fig. 14a Normalized Viewpoint

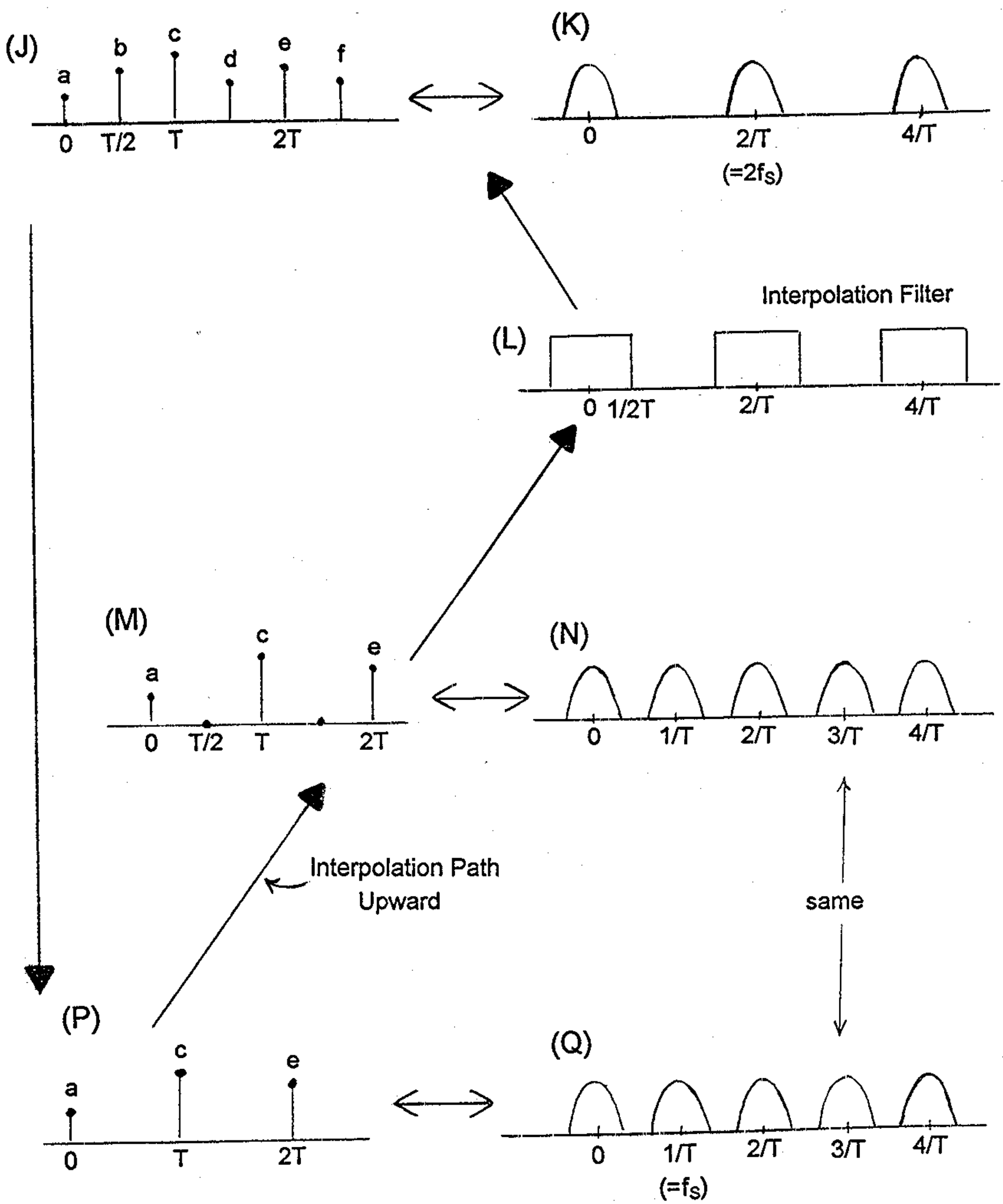



Fig. 14b Physical Viewpoint

ordinary sampling, we expect discarding half the samples to double the number of images. This is what we do see in the physical interpretation by comparing (K) to (Q). We see replicas about multiples of (2/T) in (K) and about multiples of (1/T) in (Q). This seems to be exactly right.

So what has happened in (I)? We don't see any extra sampling images. What has happened is that in the supposed implementation of the compressor (C), not only are samples discarded, but the kept samples are repositioned on the integers. This is the renormalization - understanding the current sampling rate to always be $2\pi$. This makes $2\pi$ in (I) the same as $\pi$ in (B). Note that it appears that the original spectrum in (B) is simply expanded by a factor of 2, from 0 and $2\pi$, toward $\pi$ in (I). While this sort of "rule" is sometimes put fourth, it hides what actually is going on, and just gives the wrong answer in other cases (see Section 4e). In line with equation (48), what actually happened was that the replicas were inserted, as expected, but then the whole frequency axis was expanded by a factor of two.

If the reader supposes that the physical picture is more direct and more intuitive, this is reasonable. Further, the actual idea of the compressor as a real-time physical device is clearly impossible since samples would have to appear at its output even before they arrived at the input.

## 4d-2 The Interpolation Path

As suggested above, our main study here will concern the interpolation paths, starting at the bottom with the sparser set of samples (a, c, e,....) and obtaining the denser set (a,b,c,d,e,.....) at the top. This is of course exactly what we studied as "oversampling" in Section 2e above. Thus we do not have to reintroduce the general ideas, and can concentrate here in comparing and contrasting the normalized and physical descriptions. The one thing we note is that the return paths (H) to (A) and (P) to (J) are not a simple reversal of the downsampling. We have easily thrown information away, and it will likely be more complicated getting it back. These two interpolation paths are seen to cross domains, since it is easiest to think of the setup (inserting zeros) as something that happens in the time domain, while the actual interpolation occurs through use of the interpolation filters to remove sampling images in the frequency domain.

In the normalized view, we begin with an expander (E). Like the compressor (C), this is impossible in real time. What it is supposed to do is to insert zeros between all samples at its input. Mathematically it compresses the frequency scale, (I) to (G). What it appears to do is compress the spectra from $\pi$ in (I) back toward 0 and $2\pi$, and then insert a similarly reduced replica at $\pi$ in (G). This sort of "rule" is sometimes heard, but it is misleading at best as far as a complete understanding is concerned. We certainly do not expect additional spectra replicas when no information is discarded.

In the physical view, something astounding happens when we insert zeros midway between the existing samples (P) to (M). Nothing happens to the spectra, (Q) and (N) are the same. This is less surprising when we consider that every spectral analysis tool we know of adds or integrates exponentials that are weighed by the time domain signal. If we add only time-domain zeros, we can't change the physical spectrum. The difference between (Q) and (N) is that in (Q) the sampling frequency is 1/T, while in (N) it is 2/T. That is, in putting zeros in-between existing samples (at least as place-holders for the moment) we are doubling the sampling rate. This makes perfect sense.

The remaining step, in either view, is to low-pass the zero-padded sequences to remove half the sampling replicas: [ (G) to (B) using filter (D), or (N) to (K) using filter (L)]. It is the interpolation filter that does the works of generating (perhaps recovering) the missing samples to replace the inserted zeros. We would hope that regardless of the view used, we might end up with the same filter design requirements.

Above we took some trouble to criticize the use of normalized frequencies, although acknowledging that we have to learn to use them. At that time we also mentioned that normalized frequencies in digital filter design make a lot of sense. Thus we note that the interpolation filters (D) and (L) have the same specification. The filter (D) has a low-pass cutoff of $\pi/2$ for a sampling frequency of $2\pi$. The filter (L) has a low-pass cutoff of 1/2T for a sampling frequency of 2/T. So in both cases, we are looking for a filter that has a cutoff of 1/4 the sampling rate. The considerations for the detailed specifications of this filter is a matter very similar to the oversampling case (Fig. 8).

## 4e. STUDIES WITH AID OF THE FFT

### 4e-1 Downsampling Using the FFT

It is probably clear from the discussion just above that efforts directed toward understanding rate changing can be maddening. At some point, it may be necessary to sit down and run some simulations, and it is likely the FFT (DFT) will be the tool of choice for comparing spectra. Aside from the fact that this adds yet another frequency convention, it is not a bad idea. Here we will apply the DFT (FFT) tool and first look at downsampling, as this is the case that causes the most difficulty.

Fig. 15a shows the general procedure. Here we have generated a length 64 spectrum which has a triangular shape. This can be done as a sum of triangularly weighted cosines, or by simply typing in the triangular FFT and taking the inverse FFT to get the time sequence. In either case, we want to then downsample this time sequence. In the mathematics of compression (Section 4c-3), we found it almost essential to do the downsampling with an intermediate resampling step. Thus we take the time sequence belonging to the original ("orig") spectrum, multiply every other sample by zero, and then again take the FFT. This we see as "resampled," and as expected, we see an extra spectral image. The final step is to remove the zeros from the resampled waveform, making it length 32, and taking its length 32 FFT. This is indicated as "downsampled." Now, recall that the FFT represents samples of the DTFT. That is, we look at the shape represented by the FFT spikes and not at the spikes as such. This understood, we see that the three lines of Fig. 15a correspond to (B), (G), and (I) of Fig. 14a

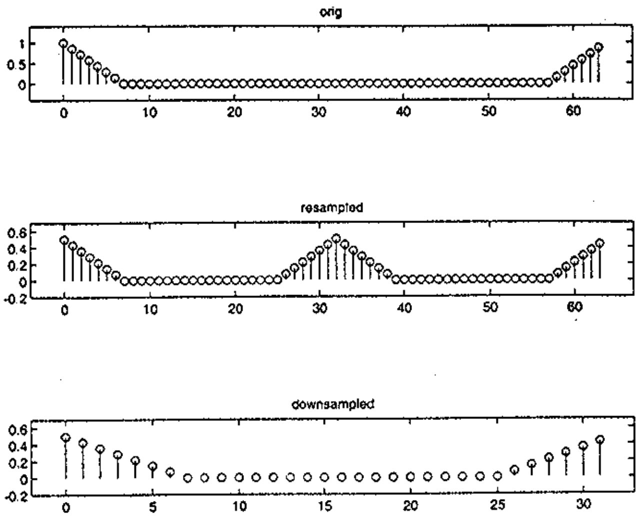



Fig. 15a Low-Pass Downsampling by 2

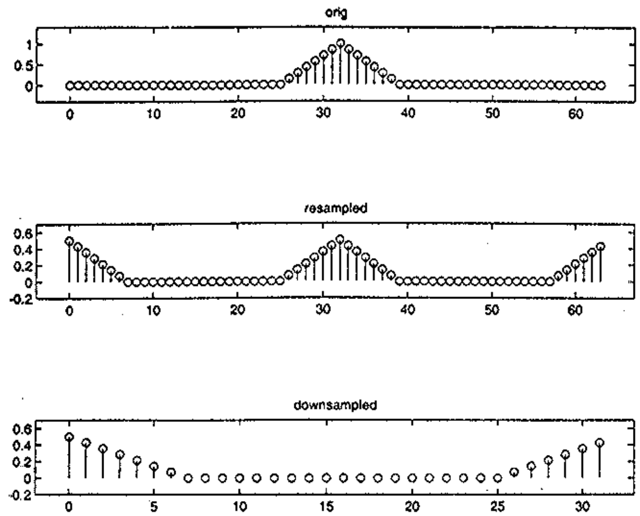



Fig. 15a **High-Pass** Downsampling by 2

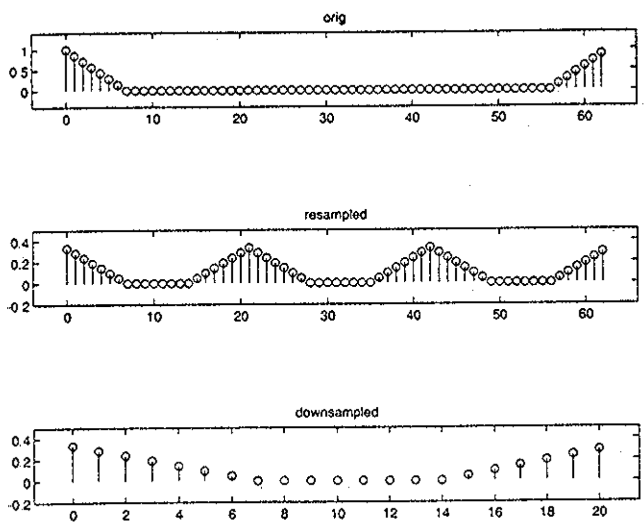



Fig. 15c Low-Pass Downsampling by 3

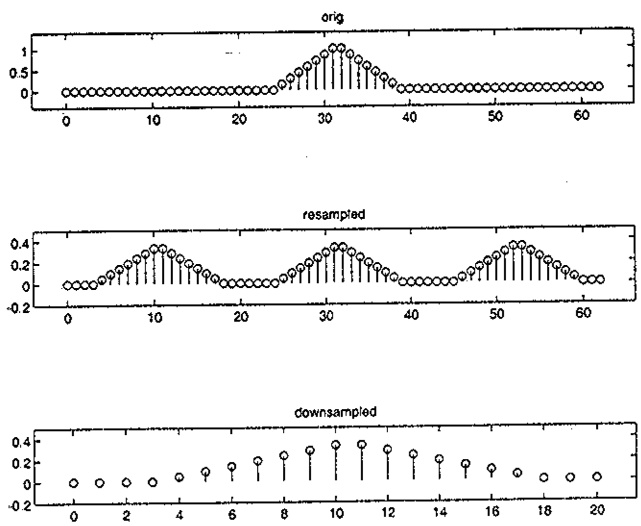
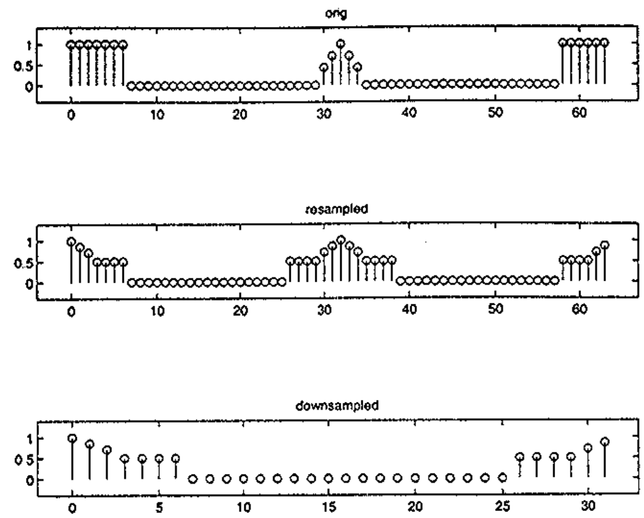
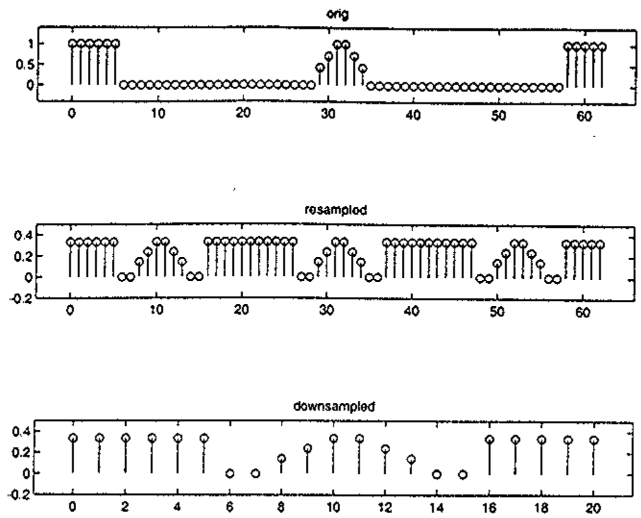



Fig. 15d **High-Pass Downsampling by 3**




<u>Fig. 15e</u> Downsampling by 2
Shows Overlap




<u>Fig. 15f</u> Downsampling by 3
Shows no Overlap

The reason why we are leery of "rules" for describing how the spectrum changes following downsampling is that there are too many individual cases. Instead we prefer the <u>procedure</u>: first resample, and then stretch. Thus for example in Fig. 15b we have a high-pass spectrum instead of the low-pass spectrum of Fig. 15a. When we resample this, we get an extra image, right in the middle as we expect. Then we stretch. Another way to look at this stretch is to say that we just keep the bottom half of the FFT. Perhaps to our surprise, the downsampled spectrum is low-pass (identical to Fig. 15a at the "resampled" and "downsampled" levels). If we were writing rules, we would have two now.

Onward to downsampling by 3. Here, in Fig. 15c, starting with a length 81 sequence that has a triangular spectrum we first resample, and we find, as expected, two new images. Note that the resampled images are always equally spaced. Resampling by N gives us N-1 additional images, N images total. Then we stretch, here looking at the lowest 27 points of the FFT. We get a low-pass result.

Fig. 15d shows a high-pass spectrum downsampled by 3. When we downsampled by 2, we got a low-pass result. What happens here? As usual, we resample, getting two additional images (again, all equally spaced when we consider the periodicity of the FFT). When we stretch, we get high-pass. Time for another rule!

Fig. 15e and Fig. 15f are shown for additional amusement. Here we combine low-pass and high-pass spectra, and use downsampling by 2 in Fig. 15e, and downsampling by 3 in Fig. 15f. Note that in Fig. 15e, the low-pass and high-pass overlap and combine. We might expect downsampling by 3 to be worse in this regard, not better. Yet Fig. 15f shows that because of the way we have chosen the original spectrum, the replicas all avoid each other.

## 4e-2 Upsampling Using the FFT

Having now employed the FFT to look at resampling and downsampling, we can look at upsampling. We recall that upsampling involves placing zeros between existing samples taken in pairs. Fig. 16a shows the case where we again start with a triangular, low-pass spectral shape (here 32 points in frequency). We then get 32 time points using the inverse FFT and insert 32 zeros between these samples. Then when we take the FFT, we get 64 frequency points in the FFT.

Perhaps to our surprise, we get a result in Fig. 16a that strongly resembles the resampled case of Fig. 15a That they should have similarities is not so surprising once we remember that both correspond to cases where zeros appear between (generally) non-zero samples. In resampling, these zeros are the result of multiplying non-zero samples by zeros. In upsampling, the zeros are simply inserted. The one difference is that upsampling results in a doubling of the number of frequency points. Fig. 16c shows a similar result for low-pass upsampling by three (compare to Fig. 15c).

Fig. 16b shows an upsampled high-pass case. We immediately understand this in terms of the compression of the frequency axis which we have discussed earlier. But this case differs from the corresponding resampling case (Fig. 15b). (Fig. 15a and 15b, the low-pass and high-pass spectra resulted in the same downsampling.) To understand this a bit better consider that a low-pass signal has, typically, consecutive samples of the same sign, while a high-pass signal tends to have samples alternating in sign. Taking every other sample, in either the low-pass or high-pass case, thus tends to favor samples of the same polarity for significant periods of time (i.e., low frequencies). But when we insert zeros, keeping all the samples, the alternating polarities of the high-frequency signal are preserved. Fig. 16d shows a high-pass signal upsampled by three. This has the same shape as the resampled by three high-pass. By an argument similar to that just used, resampling by three would tend to preserve alternating polarities (every other sample and thus every third sample tend to alternate). The high-pass nature of the spectrum (alternating signs in time) is of course preserved with the zero insertion of upsampling.

All this discussion, along with the myriad of ways for understanding the results, is of course helpful. However, one often should not resort to round-about ways of understanding as a primary means of getting the right answer. Neither is it very productive to try to establish and maintain categories, with their associated rules. What one is advised to do is work out each case presented as an individual sampling problem. Usually a sketch is most effective.

## 4e-3 Relationship Between the FFT and the DTFT Views

The reader has likely noted the strong similarity between the FFT studies and the normalized case where spectra were obtained using the DTFT. Indeed, the FFT (the DFT) is just a sampling of the DTFT. If we have a very dense set of FFT samples, we can usually infer the corresponding continuous DTFT. Alternatively, if the DTFT is sufficiently smooth or piecewise continuous (composed of straight-line segments, as is the case with our assumed

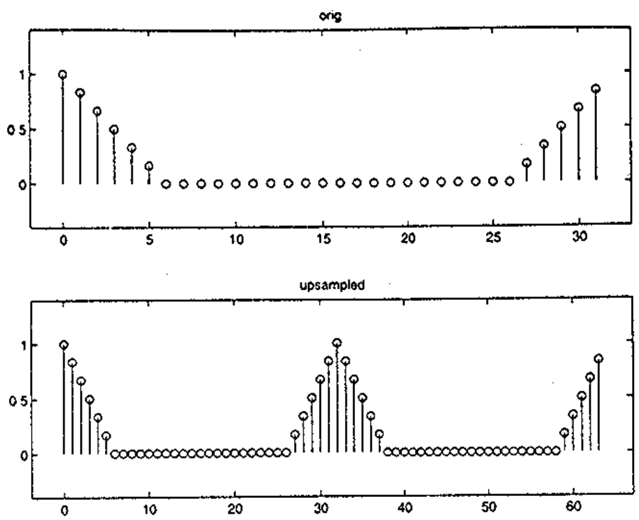



Fig. 16a Low-Pass Upsampled by 2

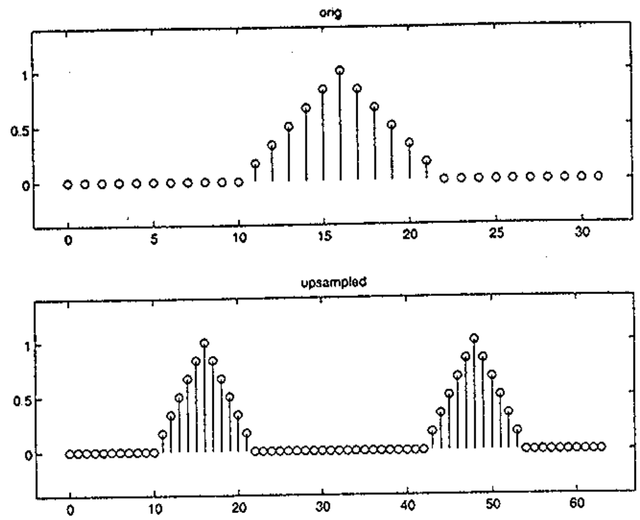



Fig. 16b High-Pass Upsampled by 2

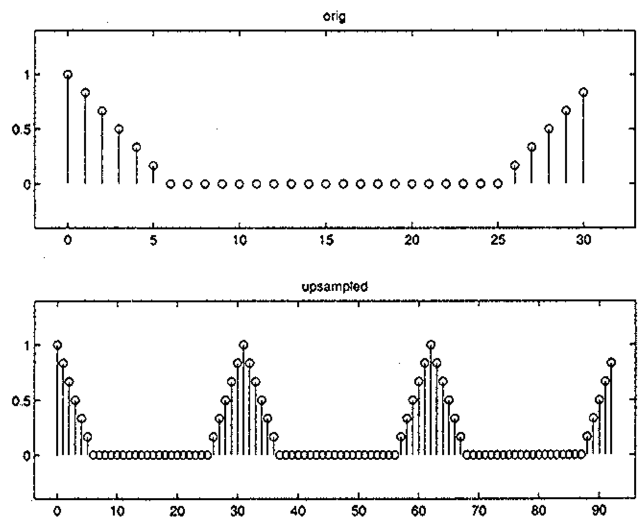



Fig. 16c Low-Pass Upsampled by 3

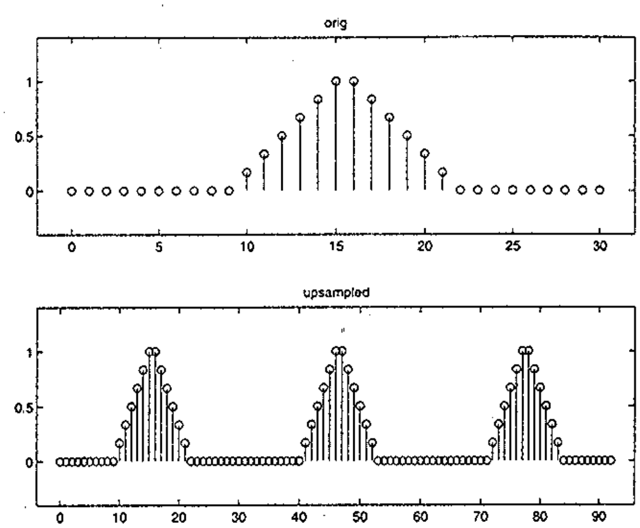



Fig. 16d High-Pass Upsampled by 3

triangle shapes for example) we can "connect the dots" of the FFT and suppose we are looking at the DTFT. The other thing we must do is stop indexing frequencies by the FFT index which runs from 0 to N-1, and consider a normalized sampling frequency of one. In an actual program, this means just dividing the frequency indices by N, and using a continuous plot rather than a discrete ("lollypop" or stem) plots. Alternatively, we just do this display mentally based on the FFT calculations.

Fig. 17a is a repeat of Fig. 15a, the low-pass spectrum resampled and then downsampled by two. These correspond to sketches G and I, coming from B, of Fig. 14a. Fig. 17b is a repeat of Fig. 16a, the upsampling of a low-pass spectrum. This corresponds to sketch G, coming from I, in Fig. 14a.

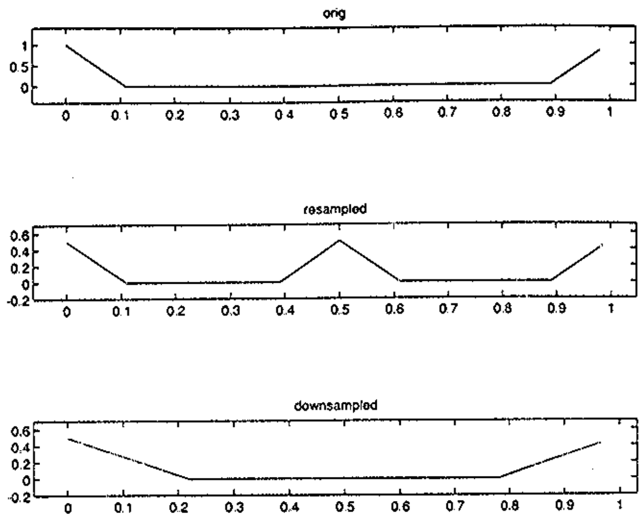



Fig. 17a Low-Pass Resampled and Downsampled by 2 (shown as DTFT) (Fig. 15a)

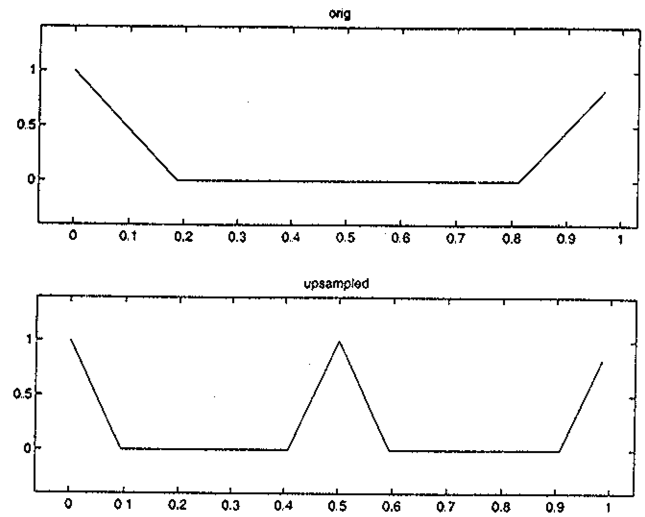



Fig. 17b Low-Pass Upsampled by 2 (shown as DTFT) (Fig. 16a)

## REFERENCES

[1] References appearing in Electronotes publications over the years: B. Hutchins,

(a) "Guide to Digital Filter Network Design - Signal Sampling and Its Consequences," Electronotes, Vol. 14, No. 144-146, pp 24-28, Dec. 1982;
(b) "Problems in the Recovery of Analog Signals from Digital Samples," Electronotes, Vol. 16, No. 169, Jan 1988;
(c) "The Importance of the Notion of Bandlimiting to Sample Reconstruction," Electronotes, Vol. 18, No. 184, Dec. 1994;
(d) "Sample Rate Changing and the Corresponding Effects on Normalized and Unnormalized Spectral Descriptions, " Electronotes, Vol. 18, No. 185, Feb. 1995;
(e) "Sampling and Recovery Based on an Average Sampling Rate," Electronotes, Vol. 19, No. 187, Aug. 1996;
(f) "Up Boxes and Down Boxes; Time and Frequency Domain; Making Sense?," Electronotes, Vol. 19, No. 188, Feb. 1997;
(g) "A Two Component Aliasing Demonstration," Electronotes Application Note AN-321, May 1992;
(h) "Mathematical Derivations of Resampling, Compression, and Expansion," Electronotes Application Note AN-346, Feb. 1998

[2] Bracewell, R.N., The Fourier Transform and Its Applications, (2nd Ed.) McGraw-Hill (1978)

ADVENTURES - Continued from page 4

located a rare book for me which he shipped to me by priority. The empty envelope arrived! At about the same time, two very large boxes of books ended up in front of our garage. They were addressed to our neighbors, and I reasoned that they were likely a collection of good physics books. We like the neighbors, so I wheelbarrowed them over.

I have seen posted comments suggesting that we do not always deliver. Sometimes they say that we do not give prompt service. These comments are justified if you suppose that delivering 99% of an order promptly is not enough, and that, with over 500 items to stock, some items need not be back ordered for 40 days or more (as stated in our literature). Frequently these sort of postings are followed up by comments from people who mention that their orders arrived promptly. If you read the archives, I think it is more than fair to say that the critics are people who have not ordered from us but who "have heard that" service is poor. (Almost without exception, we do not send materials to persons who do not send us an order.) And there are almost certainly items that are lost or miss-delivered. As always with this sort of thing, successes are seldom news. A few cases where things go wrong stand out. We all know the phenomenon.

I have also seen a couple of even more disturbing items that seem to suggest, although they are not specific (innuendo), that we keep money sent us and do not send anything. I don't recall seeing any real name associated with these remarks. Let me be very definite about this. Without going into details more than I want to, I believe that our deposit records and our shipping records are "interlocked" in a way that makes it impossible for us to deposit a check without also actually shipping. If anyone knows of a case that would even suggest otherwise, let's hear the details, and we can check our shipping records and returned items files. Did it happen to you? Did you contact us? If not, who did it happen to? Did they contact us? If you don't know the details, then you may not know anything you should be posting, as someone may well get around to asking you to produce the details! If any person insists on posting this sort of accusation without giving specifics, please do so giving your full, true name and your postal address.

And, this needs to be said as well: more than 99% of customers can be trusted. But not all. Indeed, the newsgroups often feature battles resulting from "bad trading" notices and these are often not convincing, or even counter-indicating. Mail order is a two-way street both literally and figuratively. Over the years, we have had significant losses due to payments that did not "stick." In one instance, we had to "fire" a customer and we returned his payment via the Postal Inspector, with the stipulation that he return the materials he admitted he had received. He kept the refund and the newsletters. Later, he posted a short item saying that he too had paid for something and not received it.

(3) In a couple of places I am accused of "snooping around" and even "hacking" in a search for questionable schematics. I thought materials were posted on the web to make them public. Is there a rule that you aren't supposed to look for something if the site owner doesn't want you (specifically you) to see it? I guess so. But that's beside the point.

For myself, there are three and only three ways I could end up looking at a web page. First, I could end up there by following links from a known page (ordinary surfing). Second, I often use a search engine as indeed most everyone does I believe. Third, someone may email me a link to look at. That's it. I have absolutely no idea how to hack into anyone's website. I don't recall the incident that prompted the supposed hacking, but if I found one of our schematics posted, it must actually have been public in some sense. In a way, it is flattering that anyone might suppose that I could hack (one of my TA's eyes rolled when I told him someone thought I could hack!), and I guess that if anyone still thinks so after my denial, so much the better.

(4) One very positive finding. A couple of people were worried about my health or that of my wife. When I read that, an involuntary giggle emerged. But it was certainly not a derisive giggle, but rather, an smile that accidentally overflowed. Lurking somewhere in the general diatribe was this incongruous, sincere human feeling. I was genuinely touched. In a world permeated with daily inconsequential "Howareyou-fine-andyourself-fine" here were some people who really wanted to know. Bravo - there is hope for our species.

We are well, thank you. And yourselves?

## Troubleshooting: Always a Connection

Things always seem to break and go wrong. Some people are exceptionally gifted at doing repair. By this I am not necessarily referring to a highly trained service rep who knows every detail of his or her associated products, but rather to the person who routinely just knows how to fix things: fix a car, fix a garbage disposer, debug a computer program, or even diagnose an illness. I think part of this is a matter of recognizing the interconnectivity of systems, and associating trouble in one part of the system with causes that may seem remote. Not remotely possible, just not close by in the system.

This association of a particular problem with a cause that is seemingly quite remote is something we have previously noted [1]. And the clues are usually there!

Some years back, I purchased an inexpensive AM/FM receiver, and was quite satisfied with it, and purchased several more. Right out of the box, one of them did not work. When you turned it on, the front panel was a solid yellow - no letters or numbers, and there was no sound. Of course I returned it for a working unit. Some time later, one of my receivers developed an intermittent. It would sputter and eventually the yellow screen would go blank and all sound would stop. Like most of these, it could be "repaired" with a gentle tap, or a slight twist. Clearly, there was a poor connection somewhere. It would work for a while: perhaps 10 days, or perhaps 10 minutes. But these things don't get better. Eventually you have to get around to fixing it or throwing it out. I took the cover off.

Poking around inside with a pencil eraser, I found that pressing the circuit board just about anywhere could activate/deactivate the intermittent. I wasn't getting any new information, and actually exposing the bottom of the board (not just removing the cover) looked daunting. But then I saw a red wire that did not belong there. It was soldered between a resistor lead on the top of the board and a power transistor pin. It was an after-manufacture repair! But it was also evidently well repaired.

Now I am motivated to get the board exposed, and while this is seldom easy, I do get it apart. What I find is that one of the screws that holds the board down used a hole in the heat sink as an anchor. The heat sink is attached to the power transistor in question. If, during manufacture, you don't get the heat sink completely flush before wave soldering, when you now tighten the screw, it pulls down the heat sink and thereby forces the power transistors leads further through the board, sometimes breaking the pc traces. Possibly this was a known failure mode, and since exposing the board is a lot of trouble, quite likely the manufacturer (or service) just fixed the bad one with the top-of-the-board wire. This is not unreasonable. Apparently, also at times, one trace broke and was repaired, while the next trace along the line was not completely separated, but broke later.

The point is that this took me longer than it should have. I had information that I did not use. I knew that original units could be bad in exactly the way mine eventually went bad, and I knew that this failure was apparently not uncommon (I had now seen two).

Now, recently I had a troubleshooting experience that was non-electronic but which involved our shipping system. A customer notified me that he had received his order but he thought perhaps that a couple of items were missing. Since the order was recent, I had my shipping checkoff forms readily available, and it seemed that the items he was missing were circled as sent. Further, I had packaged two "everything" orders at that time. Rarely (never before) do we get two full orders on the same day, but it did happen this time. Note that this is my first clue of something <u>unique</u> happening. Another different thing I thought of was that I had just revised the shipping form and was using the new version for the first time. Looking at the forms for the two customers, I saw that the missing volumes were circled for both customers, but with a different color for the two cases, indicating that they were in different boxes and shipped on different days. OK! Did I put them in the wrong box? So I check the postal receipts, expecting the total weight for the packages to one customer to be much greater than for the other. The individual boxes and the totals were almost identical. The totals added up to within a few ounces of each other - about 28 pounds total.

But checking other postage receipts, I saw that a full orders should have been about 32 pounds. Clearly I had messed up both orders - but how. Well, the revised form might have played some minor role in that I was circling in new places as I gathered items. But what had to be the main difficulty was doing two in parallel. In preparing two orders, I used one of the forms to gather the materials for both customers, and then I merely copied the form for the second customer. But I made a mistake in copying, and drew around three extra items. The first boxes then were shipped to both customers. The next day, in packaging the remainder, I again used only one of the two forms, and this time apparently, it was the bogus copy.

So the procedure might be something like: (1) Wow - this is weird! (2) What is unique (new, just different) about this case? (3) By what round-about unlikely path did this happen?

Do we learn? I guess not - speaking for myself. After collecting examples for years and writing about trouble-shooting, I still don't see it coming. Just a day ago my wife called me with the welcome news that the washing machine was leaking. When I looked inside the tub, I saw what was clearly not ordinary laundry. I saw it but it did not register as significant - as it should have. Of course I pulled the machine out and got the back off, and after a bit, It could be seen that water had not leaked from any particular bad connection, etc., but rather, somewhat generally over the whole tub. It came over the top. Why? Well, as far as we can tell it was because the unusual laundry was some sort of quilt material which apparently did not soak up water as fast as other laundry does. Between quilt, trapped air, and the usual water input, there was more than a tub's worth. That's all that was wrong. That which was different was significant.

In years past, we have more or less marveled at the way a remote, supposedly unrelated and unimportant change in one part of the system can cause a problem locally. The phrase "only once in a million years" comes to mind. Today however, I think that this sort of "interconnectedness" is much easier to accept. The world is more interconnected and more complex of course, but the real credit for this revelation must go to the people who write those diabolic operating systems for computers. In the good old days of DOS, when we installed a new program it might tell us to change a config file entry, or something of that nature, and then if something went wrong with an existing program, we had a fairly good idea where to look for the interference. Today automatic install wizards do the whole thing. We just sort of expect working programs to fail when we install something new. Yes - we know there is no possible connection between the two, but we are never surprised when it does happen.

Once again - something strange has happened! As I finish up this newsletter, there is some extra space. What is unique. Well, the body of the newsletter is the sampling material, and it was done on an earlier version of Word, and every time I change a period, the whole document seems to jump and transform. What to do? Well, why don't I just tell the original story. Years ago I had a stereo system and my sophomore roommate had an FM radio and I want to make his radio an FM tuner to play through my stereo amp. I had done this several times before - just put in a jack, and tap the high side of the volume control. I did this, and it worked as a tuner. But by itself, as a radio, the audio quality was highly distorted. This happened even when you disconnected the amplifier. It was obvious that my tuner modifications should not affect the radio, especially when the output cable was disconnected. What was wrong?

Well, in putting the radio back together, I noticed, but promptly forgot, that I was missing one of the three lockwashers. Missing lockwashers <u>are</u> unimportant, except, where was it. Well, the washer was steel and the radio had a speaker which has a strong magnet. The washer had jumped into the speaker, and was causing the scratchy distorted sound. I have absolutely no recollection how I found this - I do recall my roommate being impatient with my explanation that I could not have done anything to hurt his radio - so I guess I was looking hard. But I am absolutely sure it was not because I made the connection between the unique missing lockwasher and the simultaneous poor sound. These things we notice after the fact.

REFERENCES

[1] B. Hutchins, "Troubleshooting" Electronotes Application Notes 131 - 134, May /June 1979. Note that the topic discussed in this issue is essentially "Golden Rule No. 1" from AN-132. There we mentioned a problem with a stray lockwasher and a speaker magnet.

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