

ELECTRONOTES 191

NEWSLETTER OF THE
MUSICAL ENGINEERING GROUP

1016 Hanshaw Rd., Ithaca, NY 14850

Volume 19, No. 191

December 1999

GROUP ANNOUNCEMENTS

CONTENTS OF EN#191

Page 2 Analog Signal Processing
 Chapters 1 and 2
 -by Bernie Hutchins

With this issue, we begin a large series of issues that will form a comprehensive body of material on analog and digital signal processing. This series has a curious history. About a year and a half ago, the need for written material on Digital Signal Processing (DSP) at an intermediate level became apparent to me. There are more textbooks on introductory DSP than we probably need, and at the same time, we find numerous texts on specific topics in advanced DSP (e.g., statistical signal processing, multi-rate signal processing). For many years various people have noted the absence of material (or courses) at an acknowledged "intermediate" level in many different fields (the teaching of physics is perhaps an exception). Perhaps cynically viewed, one would point out that introductory text books are easier to do, have a canonically-well-defined content (often by consulting all the other texts), are easier to check (again, and perhaps with some risk, by consulting existing texts), and likely have a much larger potential market.

The lack of material at an intermediate level often leaves students without help when they want to go beyond the usual exercises and applications. At the same time, the advanced treatments may be too difficult, or more likely, will not be something one can easily pick up and utilize quickly.

Accordingly, at that time, I conceived of an idea of rewriting the basic topics of DSP with an eye to going deeper, while at the same time, providing a brief re-introduction as seemed advisable. My first topics were what I think of as the major "Elements" of DSP: Sampling, Fourier Transform (all flavors), and Filtering. The Sampling and Filtering elements have been written, and much of the material has been tested in a "third" course in DSP. Of course, I had in mind that the material

would be published in Electronotes, and the sampling element was intended for EN#191. This material will be along in the not too distant future, but first, an interesting change of plans occurred.

About 12 years ago I wrote a "text" called Analog Signal Processing which was never extensively published, and did not appear as part of Electronotes. Probably most of you have never seen it or heard of it. (A few of you have it.) In the last year or so, I have had perhaps a dozen requests for copies, even though it is not widely advertised. It seems there is still a lot of interest in Analog Signal Processing (ASP) and very few books still available. (Don Lancaster's redoubtable Active Filter Cookbook, happily, still seems to be.) To get right to the point, I decided to finally publish my ASP text in Electronotes.

The more I thought about it, the more obvious it was that I should publish the ASP material first, before the DSP. This is particularly true since it is a finished unit, and unlikely to be expanded. On the other hand, the DSP material will likely go beyond its original Sampling/Fourier/Filtering basis, for example, into multi-rate and wavelet material in the future. In the current issue, you will find the first two chapters of ASP.

ANALOG SIGNAL PROCESSING

-by Bernie Hutchins

Chapter Headings

Chapter 1	A Basis for Analog Signal Processing
Chapter 2	Active Filter Examples Leading to Complex Conjugate Poles
Chapter 3	Transfer Functions for Standard Filters
Chapter 4	Additional Filter Types: Notch and All-Pass
Chapter 5	Additional Configurations
Chapter 6	Integrator Based Designs
Chapter 7	Passive and Active Sensitivity
Chapter 8	Voltage-Controlled Filters
Chapter 9	Filtering with Analog Delay Lines
Chapter 10	Analog Adaptive Filtering

Chapters 1 through 4 are fairly traditional. In fact, the ASP collection as a whole might be considered mainly introductory through about the middle of Chapter 5, at which point some flow-graph realization methods appear in the form of the "gyrator" (inductance simulation) and the "supercapacitor" (names which have their own charm!). The beginning of Chapter 6 is perhaps again traditional, including the popular state-variable configuration, but it too then goes into a second round of flow-graph realization. None of the integrator methods will likely be totally satisfactory at higher frequencies unless some form of adjustments to allow for "active sensitivity" are included, which leads to Chapter 7.

One of the consequences of the transition from analog to digital filtering, which was very much evident in the academic community from the early 70's to the mid 80's, was that while a good portion of the very finest work in analog filtering did get into the academic journals, it was not adequately included into textbooks (fewer of which were being written), and certainly not much at all into advisories for practicing engineers. Here we have in mind mainly the flow-graph methods and the active compensation ideas. This is not to say that they were completely ignored, but just that there was not enough exposure that they automatically came to mind. Instead, we would find such as multiple-feedback infinite-gain (MFIG) configurations, often poorly implemented, on analog interface boards for DSP. In fact, many people still used the Sallen-Key configurations which were known since 1955. In all fairness, these were perfectly adequate in many applications, but not in all.

Chapter 7 involves both passive sensitivity (the R's and C's have tolerances) and active sensitivity (the op-amps are real, not ideal). This is a long chapter, and might have been split into two - one on passive sensitivity, and the other on active sensitivity. Here we have kept them together, and with good reason. You must consider both simultaneously. A structure with an excellent score in one of these may fail miserably in the other. (In fact, there are reasons to suppose that this should happen.) Chapter 7 also shows methods of correcting for active sensitivity problems.

Chapters 8 through 10 are on analog topics, but are likely (almost certainly) not found in most books on active filtering. Chapter 8 on voltage-controlled filtering (VCF) is, however, probably no surprise to long time readers of this newsletter. In fact, the appearance of the control elements in the VCF's enforce the need to consider active sensitivity, as they add phase shift which can destabilize structures. This is particularly true since the active sensitivity is expected to get worse at high frequencies, and the VCF by its very nature needs to operate over a wide range of frequencies.

Chapter 9 is a strange chapter in that it embraces ideas of discrete time - the world of DSP. This is in the guise of analog delay lines. While these analog delay lines are not used much these days, the notion of using not just "unit delays" but rather composite delays can be extremely useful in digital filtering, is practical today, and can be related to multi-rate DSP.

Finally, Chapter 10 ventures into territory not charted at all in most books on analog signal processing, although likely not totally unfamiliar to our readers. This is the area of analog adaptive filtering. A good part of the presentation does relate to the correlation cancellation loop (CCL) which is very closely related to the "LMS algorithm" of adaptive filtering in DSP. There is however, additional material relating to self-adjusting filter methods, which perhaps resemble phase-locked loops.

A couple of people who reviewed this manuscript quite a few years back found it to be "uneven" in its level of presentation, and this is probably a valid criticism if one is thinking about a text for a one-semester course. But I never thought of it just like that. In the introduction to the Musical Engineer's Handbook I quote Einstein as saying "Things should be made as simple as possible, but not simpler." This perhaps at once excuses simplified accounts and high level developments as being necessary to usefully present material that is itself useful. Moreover, I have always supposed that some readers might read parts of our material as high school students and parts as graduate students, according to needs and motivations.

CHAPTER 1

A BASIS FOR ANALOG SIGNAL PROCESSING

- 1-1 Introduction:
- 1-2 Signals and Signal Processing in General:
- 1-3 Transfer Function or System Function:
- 1-4 Laplace Transform:
- 1-5 Transfer Function by Network Analysis:
- 1-6 Frequency Response, Poles and Zeros,
and Impulse Response:
- 1-7 Operational Amplifiers

The purpose of this chapter is to introduce signal processing in general and analog signal processing in particular, and to provide review and a common basis for the material that follows. The essential ideas of transfer function, frequency response, impulse response, poles and zeros, and the like are reviewed. In particular, we are interested in obtaining a transfer function using network analysis, and then working with this transfer function to obtain the other network characterizations we find useful. We find that this can be done by "Ohm's Law" type relationships with the impedances of resistors, capacitors, and inductors. The first-order low-pass filter serves as a common example for much of what we do here. Finally, we are interested in reviewing the ideal operational amplifier or "op-amp", since this will be our main active element for the active filtering techniques that we will use.

It is possible to take a very broad view of what may be called a signal. Here however, in the context of analog signal processing, we will generally have in mind a fairly specific idea that signals are voltages or currents that are functions of time.* Further, the signals have a purpose in that they contain and carry information that is useful to us. For example, speech and music are analog signals that contain a large amount of information that is of significant importance to us. One goal of signal processing is to process a signal in such a way that the information contained is more easily extracted. Typically such a process is termed "filtering." Processes that enhance useful information, or that reject unwanted information, can be considered as filters. In this sense (signal processing as filtering), we have a result that is generally consistent with everyday experience: that filters improve the quality or the purity of something that is of use to us.

While filtering is a major aspect of signal processing, some related areas are signal analysis, signal synthesis, and signal detection. In signal analysis, we are concerned with the way a signal may be represented as component parts. Signal analysis in terms of a frequency spectrum is perhaps most familiar. Signal synthesis may be as simple as waveshaping of one waveform to another, but it can also involve complicated modelings, as for example, when a certain combination of filters is used to synthesize artificial speech. Signal detection often involves initial enhancement with filtering, but specific filters that look for a specific signal element may also become involved.

In general, signal processing will involve two or more signals, typically considered in a "before and after" or input/output relationship. In such a case, the signals are related in terms of a connecting system, such as is suggested in Fig. 1-1. We will consider that the system alters the input, thus producing the output. Two complimentary points of view, one of analysis, and the other of synthesis, are commonly encountered with this system approach.

*We must always be aware that sequences of numbers, discrete in time, are also signals. These are the signals of digital filters, and are therefore of very great importance. Consideration of a possible "digital filter alternative" is a logical, useful, and necessary step in an approach to any signal processing problem, including those that seem inherently analog in nature.

From the analysis point of view, we would consider the system as given, and we need to determine its effect on a known input signal. In such a case (a communication link for example), it would be necessary to first analyze the system, then determine the output from the known input, and ultimately consider if the output is satisfactory, or if some corrective measures must be employed.

From a synthesis point of view, which essentially represents a filter design problem, the input signal and some notion of an acceptable output signal are known to us. The problem is to design a satisfactory system (filter) to put the output signal within acceptable bounds. For example, we might want to reduce the level of "AC hum" from a speech signal by at least 40 db.

Common to both the analysis and the synthesis cases is the need to determine how a given system can be analyzed, and how the input and output are related by the system. Here the notion of a transfer function for a linear system is of great value.

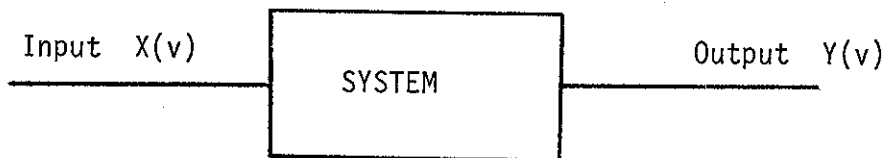


Fig. 1-1 General Input/Output Concept of a System

1-3 TRANSFER FUNCTION OR SYSTEM FUNCTION:

In general, it is convenient to have a straightforward characterization procedure for a system that describes the input/output relationship for all inputs and outputs of practical interest. Such a relationship is called a transfer function (or system function) of the system. If the input can be described by $X_k(v)$ and the corresponding output by $Y_k(v)$, then we would like a transfer function $T(v)$ given by:

$$T(v) = Y_k(v)/X_k(v) \quad (1-1)$$

which would be true for all $Y_k(v)$ and $X_k(v)$ of interest. Under what conditions would such a $T(v)$ be obtainable, what is the variable v , and how do we obtain $T(v)$?

In our signal processing work, we must always be aware of the usefulness of describing signals both in the frequency domain and in the time domain. Indeed, many filter problems are posed primarily in terms of a needed frequency response (such as asking for a low-pass filter). Thus in looking for a variable for a transfer function, time and frequency are obvious alternatives, and we are familiar with the Fourier and Laplace relationships between the time domain and the frequency domain.

If we choose time t as the variable v in equation (1-1), then it is easy to see that $T(v) = T(t)$ is time-varying unless $X_k(t)$ and $Y_k(t)$ are simply scaled versions of each other. Of more use is the choice of v as a frequency. Below we will find the usefulness of the Laplace variable s for frequency, in which case a transfer function $T(s)$ becomes:

$$T(s) = Y(s)/X(s) \quad (1-2)$$

This gives a time-invariant $T(s)$, which is not only simpler to use, but consistent with our intentions to employ constant valued resistors, capacitors, and inductors in our filters. Also, $T(s)$ in equation (1-2) is in a form that is a promising step toward a frequency response function, since it is the ratio of two functions of frequency.

Perhaps the greatest virtue of $T(s)$ as a transfer function is that we will find below that $T(s)$ can be obtained from usual notions of network analysis. Given a network, we can determine $T(s)$ from circuit laws using only algebra. From this, we can obtain the frequency and phase responses, and if desired, the impulse response of the system as the inverse Laplace transform of $T(s)$.

1-4 THE LAPLACE TRANSFORM:

The Laplace transform of a signal $f(t)$ is given by:

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt \quad (1-3)$$

While the Laplace transform and its inverse are fundamental to most of what we do in analog signal processing, relatively speaking we have little direct use for the Laplace transform equations. However, in reducing differential equations to algebraic ones, the Laplace transform offers us the algebraic transfer function relationship of equation (1-2). Accordingly we shall find great use for the Laplace transform variable s , even though equation (1-3) is not often used.

According to our interpretation of $T(s)$ as $X(s)/Y(s)$, all of $T(s)$, $Y(s)$, and $X(s)$ are Laplace transforms as given by equation (1-3). Clearly the corresponding inverse transforms, $x(t)$ of $X(s)$ and $y(t)$ of $Y(s)$ are the time functions of the input and output respectively. Although we seldom need to obtain $T(s)$ except by network analysis, it is possible to obtain $T(s)$ as the Laplace transform of $g(t)$, where $g(t)$ is the impulse response of the system. This can be seen since the Laplace transform of an impulse input is $X(s) = 1$, from which we see from equation (1-2) that the response to this impulse, the impulse response, is $Y(s) = T(s)$ in the Laplace transformed (frequency) domain. Since it is usually easiest to obtain $T(s)$ from network analysis, we do on occasion find it useful to obtain $g(t)$ using the inverse Laplace transform (or more likely, suitable tables).

The advantage obtained by using the s -domain (the algebraic transfer function relationship) must be given back if we return to the time domain. This results in a convolution relationship by which the output time waveform is the input time waveform convolved with the impulse response $g(t)$ of the system. This is our usual notion that multiplication in one domain corresponds to convolution in the other domain.

Inherent to the use of Laplace transform is the idea that we are dealing with linear systems. That is, the response of the system to the sum of two inputs is the sum of the responses that would be obtained if the inputs were individually applied. Thus in total we are concerned with linear, time-invariant systems. We are fortunate in analog signal processing that most of the systems we need to become involved with yield well to this analysis, although at times non-linearities due to non-ideal system elements may provide interesting complications.

We become very much accustomed to working with the Laplace transform variable s ; so much so that we can often omit the (s) functional notation. In general, we shall use capital letters for Laplace transformed quantities, and omit the (s) , except for $T(s)$. The corresponding time functions will be denoted by lower case letters, and usually will include the (t) functional notation.

1-5 TRANSFER FUNCTIONS BY NETWORK ANALYSIS

Above we have suggested that we hope to obtain $T(s)$ by some relatively simple straightforward procedure, and specifically, by network analysis. In order to see how this can be done, we need to see how the Laplace transformation applies to each of the circuit elements we expect to encounter in analog signal processing. These are the resistor, the capacitor, the inductor, the linear amplifier, and the analog delay line. For each of these, we need to obtain an "Ohm's Law-Like" or other simple s -domain notation.

The resistor is governed by Ohm's law in the time domain:

$$R = v(t)/i(t) \quad (1-4)$$

and this impedance remains unchanged in the s-domain, and can be denoted as:

$$Z_R(s) = R = V(s)/I(s) \quad (1-5)$$

On the other hand, the capacitor (C) is known to obey a charge(q) to voltage(v) relationship:

$$q(t) = C v(t) \quad (1-6)$$

or, differentiating charge to current:

$$i(t) = dq(t)/dt = C dv(t)/dt \quad (1-7)$$

In the s-domain, the derivative becomes a power of s, so equation (1-7) becomes:

$$I(s) = C s V(s) \quad (1-8)$$

which corresponds to an impedance relationship:

$$Z_C(s) = 1/sC = V(s)/I(s) \quad (1-9)$$

In a similar manner, the inductor (L) is known in the time domain to obey the law:

$$v(t) = L di(t)/dt \quad (1-10)$$

so in the s-domain we have:

$$V(s) = L s I(s) \quad (1-11)$$

for an impedance relationship:

$$Z_L(s) = sL = V(s)/I(s) \quad (1-12)$$

Thus we have obtained three "Ohm's Law-Like" relationships, equations (1-5), (1-9), and (1-12) which apply in the s-domain.

We can deal with the analog time delay of delay T by applying equation (1-3) directly, in which case we can show that:

$$V_{out}(s) = e^{-sT} V_{in}(s) \quad (1-13)$$

so passing through a delay T is equivalent to multiplying by e^{-sT} . Finally, the linear amplifier of gain K simply scales the Laplace transform as:

$$V_{out}(s) = K V_{in}(s) \quad (1-14)$$

We will not need relationship (1-13) until Chapter 9, but those familiar with digital filters will recognize this as the z^{-1} notation used for a delay there. For easy reference, the essentials of the s-domain representation are listed in Fig. 1-2.

What we are going to do as we move to active filtering is to first look at the way that we can use network analysis on passive networks consisting of R's, L's, and C's. Then we will introduce the finite gain amplifier K, and show how we can use it to get rid of the inductors L, which are often too large or too heavy at frequencies of interest. Basically all that we will do is to apply the "Ohm's Law" relationships to the circuit voltages and elements in the s-domain. Then as is necessary, we will use Kirchhoff's current conservation law at unknown nodes. Some examples will follow.

EXAMPLE 1-1 Find the transfer function $T(s)$ of the simple RC network of Fig. 1-3.

Here we assume that the voltage $V_{in}(s)$ is supplied by a zero output impedance source, and that there is no loading at $V_{out}(s)$. Consequently we can see that what remains is a simple voltage divider for $T(s) = V_{out}(s)/V_{in}(s)$, and the problem is

Fig. 1-2 ANALOG CIRCUIT ELEMENTS IN THE LAPLACE TRANSFORM DOMAIN

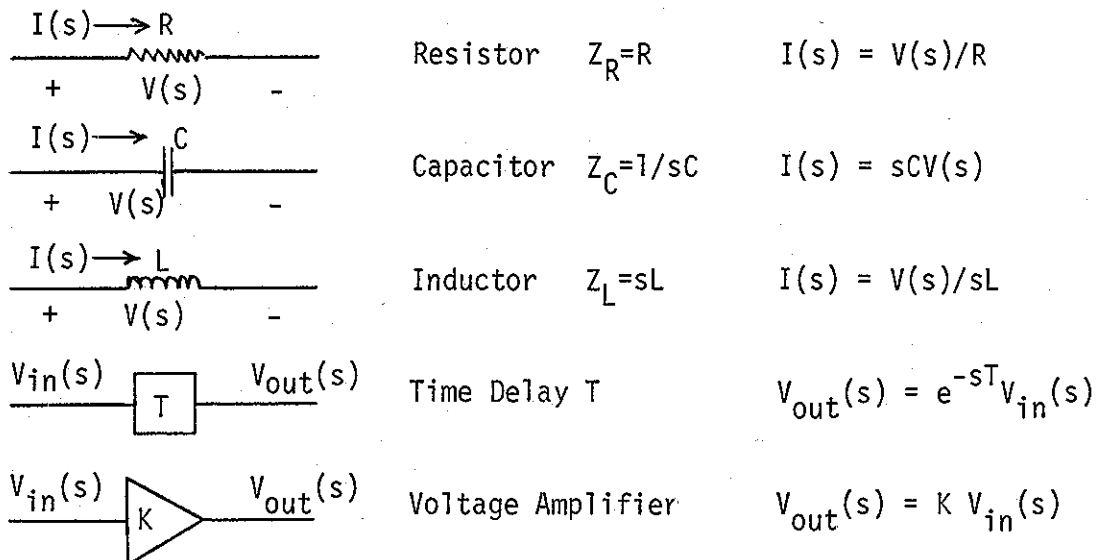
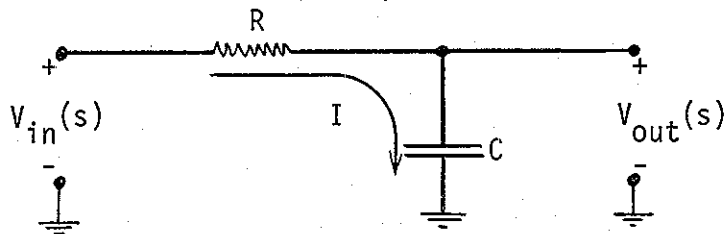


Fig. 1-3
Simple RC
Circuit



just to recognize that the lower leg of the divider, the capacitor, is equivalent to a "resistor" of "resistance" (impedance) $1/sC$. Accordingly we can just write down:

$$T(s) = \frac{1/sC}{R + 1/sC} = \frac{1}{1 + sCR} \quad (1-15)$$

Note that you can also solve the problem by recognizing that there is one and only one current I through the two elements.

In this first example, we have been careful to note that the voltages are functions of s , and that they are referenced to ground. In general, we will be able to simplify diagrams and notation by assuming that the node voltages are all in the s -domain, and that they are referenced to ground.

EXAMPLE 1-2 Find the transfer function $T(s)$ of the RLC circuit of Fig. 1-4.

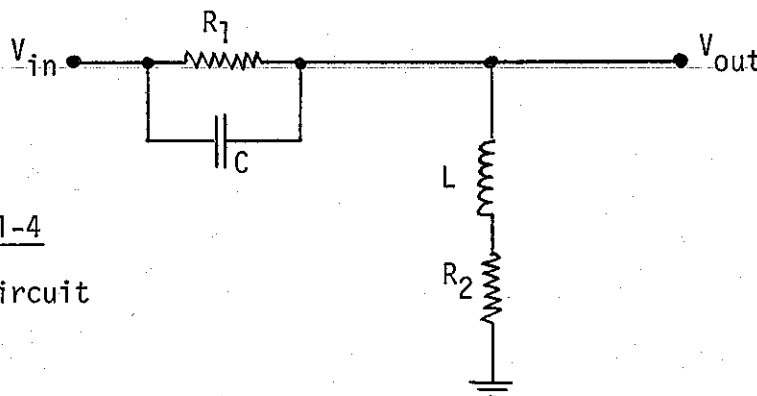


Fig. 1-4
RLC Circuit

There are several ways of solving this network, including a voltage divider approach based on parallel and series impedances in the legs. Here however we can use the method of just summing currents at the output node. Note that according to our assumptions, no current flows from the output node. Current conservation then gives the following equation:

$$\frac{V_{in} - V_{out}}{R_1} + \frac{V_{in} - V_{out}}{1/sC} = \frac{V_{out}}{R_2 + sL} \quad (1-16)$$

This involves only V_{in} and V_{out} along with the passive elements, and can be solved for $T(s)$ as:

$$T(s) = \frac{s^2 + s\left[\frac{1}{CR_1} + \frac{R_2}{L}\right] + \frac{R_2}{LCR_1}}{s^2 + s\left[\frac{1}{CR_1} + \frac{R_2}{L}\right] + \frac{R_1 + R_2}{LCR_1}} \quad (1-17)$$

This is obviously a somewhat more interesting equation than the simple one of equation (1-15).

Note that in setting up the current equation (1-16) we assumed that current flow from + to -. If we happen to choose a different polarity, the current is reversed, a sign changes in the sum, and everything still comes out the same in the end. More will be said about selecting current directions in the examples of Chapter 2.

Another point that can be made here is that we of course wonder if equation (1-17) is correct, or if there may be some mistake. One check that we can make, which won't prove we are right, is to check dimensions. If a dimensional error is found, we surely have something wrong. In fact, it is good practice to check dimensions not just at the end, but as each and every term is written down. With some practice and experience, dimensional checking becomes automatic, and very productive at early detection and subsequent correction of algebraic errors. How do we do this?

$T(s)$ is a ratio of voltages, and must therefore be dimensionless. In fact, we see in equation (1-17) that it is the ratio of frequency-squared terms, judging by the leading s^2 terms. All the remaining terms should be squared frequencies, or equivalently, inverse time squared. Now, we know that an RC product is an RC "time constant" so the dimensions of RC are inverse frequency. Later we will find this sufficient, since we will be doing mainly RC active filters. However, for the moment, can we also check the inductance? We probably remember the RCL resonance circuit from physics where the product LC is time squared. If this is true, then $R/L = RC/LC$ must be inverse time, or frequency. This done, we see that the dimensions of equation (1-17) are all correct frequency-squared terms, top and bottom.

One final warning about dimensional checking can be made. It is not unusual to encounter a procedure called "normalization" that can be useful at times. This involves setting some of the component values to 1 or to other simple numbers, thereby simplifying algebra. However, this can make dimensional checking much more difficult. Thus while algebra may be simpler, you give up an important checking tool.

Many more network analysis examples could be given. In particular, we have not used our voltage amplifier in an example yet. This will come in Chapter 2.

1-6 FREQUENCY RESPONSE, POLES AND ZEROS, IMPULSE RESPONSE

As mentioned above, often we are interested in knowing how much a given system enhances or rejects signals of various frequency at the input. This is what we call the frequency response of the system. So far, we have arrived at a notion of a transfer function, and a means of obtaining the transfer function. We saw that the transfer function looked like a step in the right direction for obtaining a frequency response, since it was a ratio of output to input voltages, and was a function of frequency (complex frequency s in this case). We will shortly show that the frequency response is given by the magnitude of the transfer function $T(s)$. Thus we

will find it convenient to just use the notation $|T(s)|$ for frequency response. It will be further understood that we are to evaluate $|T(s)|$ only on the $j\omega$ -axis of the s -plane, as will be discussed below. Since $T(s)$ is a complex function, we find its magnitude by multiplying the function by its complex conjugate and then taking the square root. Thus $|T(s)|$ evaluated at $s = j\omega$ is given as:

$$|T(s)| = [T(j\omega) \cdot T(-j\omega)]^{1/2} \quad (1-18)$$

This gives us a means of going from a network, to a transfer function $T(s)$, and ultimately to a frequency response $|T(s)|$. However, we need to see how this relates to our laboratory notion of frequency response. In the lab, we measure a frequency response by applying sinusoids of different frequency to the input of a system, and then measuring the output amplitude relative to the input amplitude. This seems to us to be a time-domain process rather than a frequency-domain process. However, we must keep in mind that we always do this with sinusoids, and we always wait for the transient to die down (although in fact we don't usually make an issue of this). We will now show that this apparently time-domain measurement - the ratio of the amplitude of output and input sinusoids - is the same as the calculation of equation (1-18).

Fig. 1-5 Frequency Response "Test"

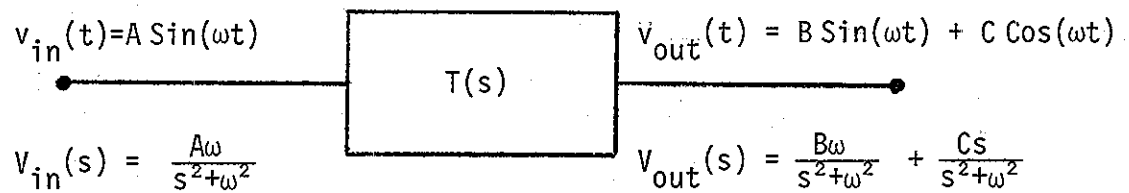


Fig. 1-5 shows a system $T(s)$ being subjected to a frequency response "test". $T(s)$ is a linear system with a sinusoidal input, and the output must be a sinusoidal of the same frequency, although the phase and amplitude will in general be changed across the system. We can choose as the input test signal:

$$v_{in}(t) = A \sin(\omega t) \quad (1-19)$$

The output can be represented in general as a sinusoidal with a different amplitude A' and a different phase ϕ :

$$v_{out}(t) = A' \sin(\omega t + \phi) \quad (1-20)$$

However we can also write $v_{out}(t)$ in terms of sine and cosine components as:

$$v_{out}(t) = B \sin(\omega t) + C \cos(\omega t) \quad (1-21)$$

[Here as a matter of simple trig identities, $B = A' \cos \phi$, $C = A' \sin \phi$, $A' = \sqrt{B^2 + C^2}$, and $\phi = \tan^{-1}(C/B)$.] Since the sine and cosine components are 90° out of phase, the output amplitude divided by the input amplitude, A'/A , our measured frequency response, is given by:

$$|T(s)|_{\text{measured}} = \frac{\sqrt{B^2 + C^2}}{A} \quad (1-22)$$

What we have guaranteed about $T(s)$ is that it is the ratio of the Laplace transform of the output to the Laplace transform of the input. Consulting tables (see end of chapter or other source), the Laplace transform of the input $v_{in}(t)$ is:

$$V_{in}(s) = A\omega/(s^2 + \omega^2) \quad (1-23)$$

while the Laplace transform of the output, equation (1-21) is:

$$V_{out}(s) = B\omega/(s^2+\omega^2) + Cs/(s^2+\omega^2) \quad (1-24)$$

Since we agree that $T(s)$ is $V_{out}(s)/V_{in}(s)$, using equations (1-23) and (1-24), we have:

$$T(s) = \frac{B\omega + Cs}{A\omega} \quad (1-25)$$

If we take the magnitude of $T(s)$ using equation (1-18), we obtain:

$$|T(s)| = \frac{\sqrt{B^2+C^2}}{A} \quad (1-26)$$

which is the same as equation (1-22).

The significance of the above is that if we desire to know the frequency response, as the ratio of output to input sinusoidal amplitudes, we can obtain it by first obtaining $T(s)$ using network analysis, and then by simply solving for $|T(s)|$ using equation (1-18). Later we shall find additional ways of obtaining the frequency response in cases where equation (1-18) is algebraically inconvenient.

We are also at times interested in the phase response, and not just the magnitude response of $T(s)$. The phase response can be obtained as:

$$\phi(\omega) = \tan^{-1} \frac{\text{Im}\{T(j\omega)\}}{\text{Re}\{T(j\omega)\}} \quad (1-27)$$

This can easily be developed by considering that from equations (1-20) and (1-21) it was seen that:

$$\phi(\omega) = \tan^{-1}(C/B) \quad (1-28)$$

from which equation (1-27) results from use of equation (1-25). Thus to find the phase response we first obtain $T(s)$, then we take $T(j\omega)$ and find its real and imaginary parts, and then take the inverse tangent of the ratio of the imaginary part to the real part. Note that by "imaginary part" we do not mean to include the j , but rather the real number that multiplies the j . For example, the imaginary part of $6 + 14j$ is 14, and not $14j$.

EXAMPLE 1-3 Find the frequency response and phase response of the RC low-pass of Fig. 1-3, Example 1-1.

We have already determined $T(s) = 1/(1+sCR)$ so $T(j\omega) = 1/(1+j\omega RC)$ and:

$$|T(s)| = \left[\frac{1}{1 + j\omega RC} \cdot \frac{1}{1 - j\omega RC} \right]^{1/2} = \left[\frac{1}{1 + \omega^2 R^2 C^2} \right]^{1/2} \quad (1-29)$$

To find the phase response, put $T(j\omega)$ in the form:

$$T(j\omega) = \left[\frac{1}{1 + j\omega RC} \cdot \frac{1 - j\omega RC}{1 - j\omega RC} \right] = \frac{1 - j\omega RC}{1 + \omega^2 R^2 C^2} \quad (1-30)$$

from which equation (1-27) gives:

$$\phi(\omega) = \tan^{-1}(-\omega RC) = -\tan^{-1}(\omega RC) \quad (1-31)$$

Fig. 1-6 shows plots of the magnitude response and the phase response corresponding to equations (1-29) and (1-31) for the case $RC=1$, and plotted for $\omega = 0$ to 10. Note that at $\omega = 1$, the magnitude response is $1/\sqrt{2}$ while the phase response is -45° . As the frequency gets higher and higher, the magnitude response approaches 0 while the phase response approaches -90° .

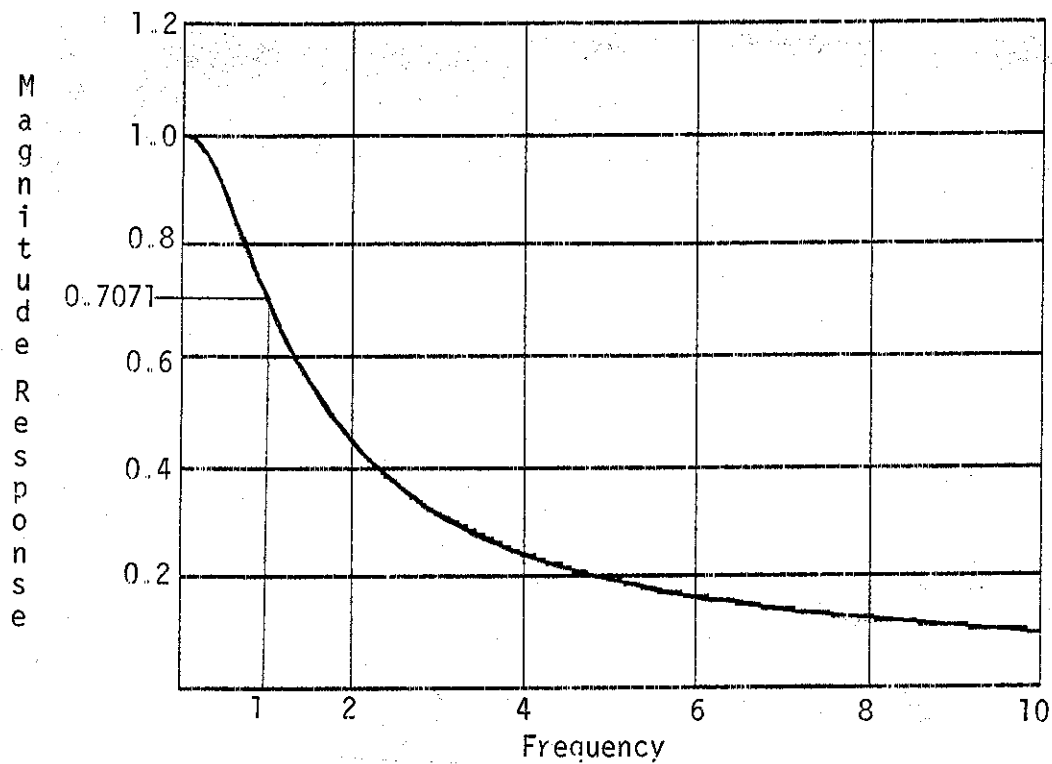


Fig. 1-6a

Magnitude of Frequency Response of First-Order Low-Pass

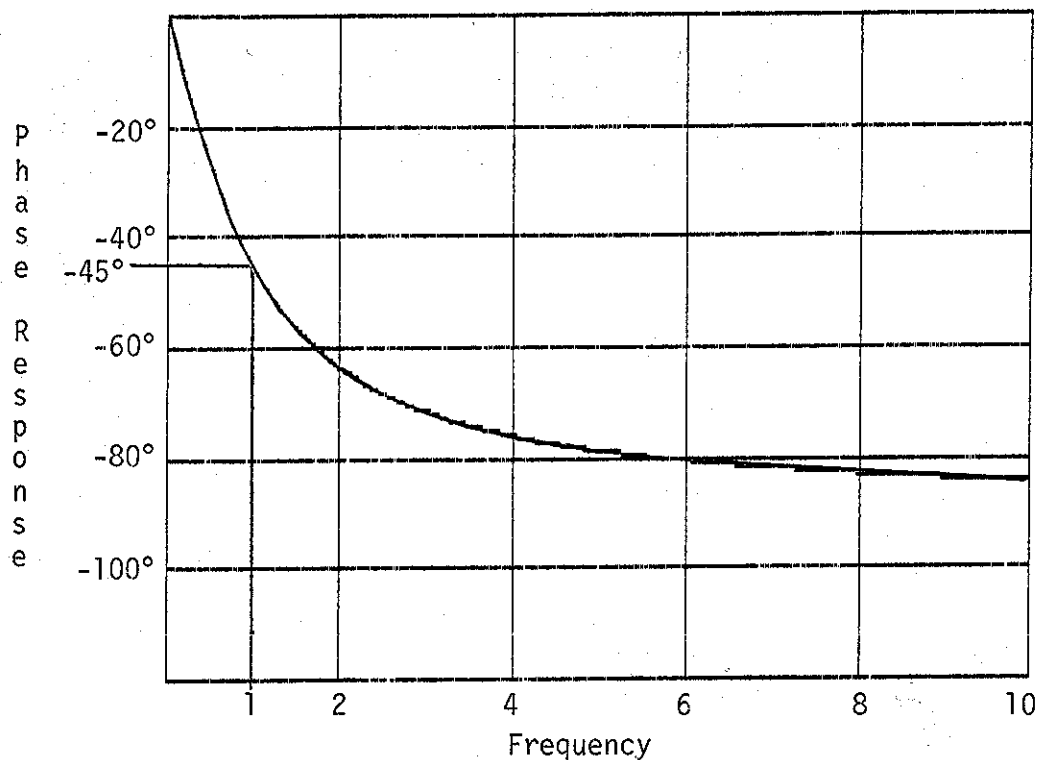


Fig. 1-6b

Phase Response of First-Order Low-Pass

In addition to using $T(s)$ directly to relate Laplace transforms, and to obtaining the frequency and phase response from $T(s)$, we also understand our system in terms of its poles and zeros, as they occur in the s -plane. Specifically we are interested in transfer functions $T(s)$ that are ratios of polynomials in s , which comes naturally from our use of realizable (or already realized - to be analyzed) networks. For analog systems we have a numerator polynomial of an order that does not exceed the order of the denominator polynomial. These polynomials in s can be factored down to first-order terms, and from these, the roots of the polynomial are obtained. The roots are the values of s for which the polynomial becomes zero. For $T(s)$, the roots of the numerator are called the zeros of the

network while the roots of the denominator are called the poles of the network. It is evident that if the numerator becomes zero, $T(s)$ itself must also become zero. In the case of a pole, the denominator becomes zero, and $T(s)$ blows up.

Probably no single piece of information about a system is of more significance than the positions of its poles. Note that the roots of a polynomial in s are in general complex numbers in a complex "s-plane." The Laplace variable, or complex frequency s , in general has a real part σ and an imaginary part ω , or:*

$$s = \sigma + j\omega \quad (1-32)$$

Thus we have the notion of a complex number plane for s , the so-called "s-plane." We can discuss the location of the poles and zeros of a system in the s-plane with particular regard for their proximity to the $j\omega$ -axis, the line in the s-plane where we evaluate the frequency response [see equation (1-18)].

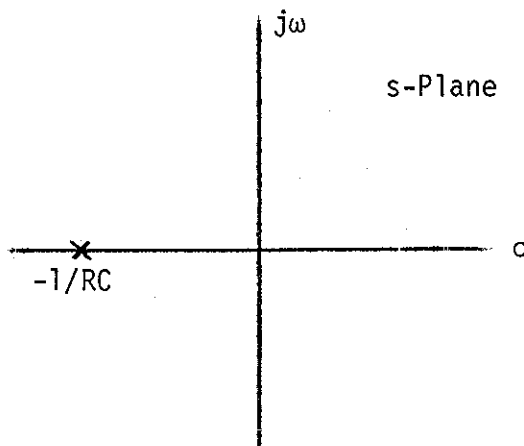
It is probably well-known to the reader from linear system theory that poles of stable systems must be in the left half of the s-plane. That is, if $\sigma_p + j\omega_p$ is a pole of $T(s)$, then $T(s)$ being stable implies that σ_p is negative. This is easily shown by considering that a positive σ_p will result in an exponentially growing component in the inverse Laplace transform of $T(s)$ (see problems at end of chapter). Poles on the $j\omega$ -axis, incidentally, correspond to oscillators as long as the poles are only first-order.

EXAMPLE 1-4 Find the poles and zeros of the RC low-pass of Fig. 1-3, Example 1-1, and plot them in the s-plane.

Since $T(s) = 1/(1+sCR)$, there are no zeros in this system, since there are no values of s that make the numerator go to zero (the numerator is always 1). The denominator becomes zero when $s = -1/RC$, so there is a real pole at $-1/RC$. Fig. 1-7 shows the pole plotted in the s-plane. We note that the pole has a negative real part and therefore, the system is stable.

Fig. 1-7

Pole/Zero plot of RC low-pass shows pole at $-1/RC$, indicated by x mark. There are no zeros. A zero would be indicated as a circle.



This simple example has only one real pole. In general, a system would have complex poles in complex conjugate pairs, and would have zeros as well. Poles are indicated by x marks, and zeros are indicated by small circles. For stability, poles are in the left half-plane, but zeros could appear anywhere in the s-plane.

*Note the incongruous notation in that frequency that is real in the laboratory sense (ω) is imaginary in the Laplace sense. Actually, very likely the most familiar and therefore the most "real" frequency of all is probably frequency in Hertz (Hz, of formerly, cycles-per-second), which we denote f . Radial frequency ω is related to f by $\omega = 2\pi f$, and has units of radians-per-second (which is numerically larger than f in Hz). It is well to be aware that errors of 2π are quite common when doing actual calculations, and results that seem to be off by about a factor of 6 should be investigated for such 2π errors. The real part of s (σ) is not related to the oscillatory aspect of complex frequency, but rather to the growth or decay of amplitude of a signal.

A better understanding of how pole/zero plots are related to the frequency response can be had by considering a more general case of a transfer function:

$$T(s) = \frac{b_0 + b_1s + b_2s^2 + \dots + b_Ms^M}{a_0 + a_1s + a_2s^2 + \dots + a_Ns^N} \quad (1-33)$$

where M does not exceed N. $T(s)$ can be factored down to first-order terms. If M or N is greater than two, this can be done numerically by computer and is called "root finding" or "pole/zero factoring." Accordingly we can obtain a revised form of equation (1-33) as:

$$T(s) = A \frac{(s-z_1)(s-z_2) \cdot \dots \cdot (s-z_M)}{(s-p_1)(s-p_2) \cdot \dots \cdot (s-p_N)} \quad (1-34)$$

Where the z_m are the zeros, the p_n are the poles, and A is an overall multiplier. [As a corollary to equation (1-34), note that knowing all poles and zeros gives the transfer function up to within an arbitrary multiplicative constant.] It is a simple step now to place magnitude bars on equation (1-34) and arrive at an equation for the frequency response $|T(s)|$, alternative to equation (1-18), as:

$$|T(s)| = \frac{|A| |s-z_1| |s-z_2| \cdot \dots \cdot |s-z_M|}{|s-p_1| |s-p_2| \cdot \dots \cdot |s-p_N|} \quad (1-35)$$

The interpretation of equation (1-35) is that the magnitudes are the distances, in the complex s -plane, from the poles or zero, to the point s of interest. For the frequency response, this is a point on the $j\omega$ -axis where ω is the frequency at which the frequency response is to be determined. Accordingly we see that the frequency response is proportional to the product of the distances to the zeros and inversely proportional to the product of the distances to the poles. The distances could even be determined from actual measurement with a ruler on an accurate s -plane plot, and this general "geometric interpretation" of frequency response is extremely useful for estimating and sketching purposes. It can also lead to a numerical method of evaluation. Each of the distances in equation (1-35) is of the form $|s-x|$, where x is a pole or zero with real part r and imaginary part i . For such a point x , the distance to a point on the $j\omega$ -axis is simply:

$$|s-x| = [(\omega-i)^2 + r^2]^{1/2} \quad (1-36)$$

and this can be used for each term of equation (1-35).

EXAMPLE 1-5 Geometrically determine the frequency response of the RC low-pass of Fig. 1-3, Example 1-1, from its pole/zero plot.

Fig. 1-8 shows the pole plot as found in example 1-4. The frequency response is inversely proportional to the distance ρ from the pole at $-1/RC$, and this gives:

$$|T(s)| = \frac{1}{[\omega^2 + (1/RC)^2]^{1/2}} \quad (1-37)$$

which is in agreement with equation (1-29) to within an arbitrary constant.

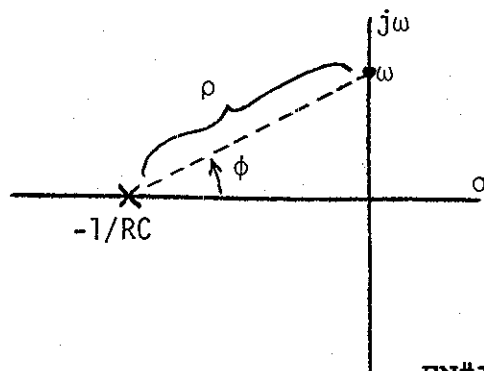


Fig. 1-8

Frequency Response from Pole/Zero Plot

Note that it is clear from this viewpoint why the response for $\omega = 1/RC$ is down by $1/\sqrt{2}$ from the dc value. Further note that the phase can be interpreted as the angle ϕ in Fig. 1-8

Since equation (1-18) for $|T(s)|$ is algebraically inconvenient at high order, and since equation (1-35) is only convenient when all the poles and zeros are known, we need yet a third equation when $T(s)$ is unfactored as in equation (1-33). Such a method is available if we write:

$$|T(s)| = |N(s)| / |D(s)| \quad (1-38)$$

where the numerator $N(s)$ and the denominator $D(s)$ are unfactored polynomials. Both $N(s)$ and $D(s)$ are complex numbers in general, and their magnitudes can be obtained by taking the square root of the sum of the square of the real part plus the square of the imaginary part. The problem is thus simple if we can isolate the real and the imaginary part of the polynomials. This can be done for $T(j\omega)$, as we can show for example for $D(s) = D(j\omega)$ as:

$$D(s) = a_0 + a_1 s + a_2 s^2 + \dots + a_N s^N \quad (1-39)$$

$$D(j\omega) = a_0 + ja_1\omega - a_2\omega^2 - ja_3\omega^3 + a_4\omega^4 + \dots + j^N a_N \omega^N \quad (1-40)$$

$$\begin{aligned} &= [a_0 - a_2\omega^2 + a_4\omega^4 - a_6\omega^6 + \dots] \\ &\quad + j[a_1\omega - a_3\omega^3 + a_5\omega^5 - a_7\omega^7 + \dots] \end{aligned} \quad (1-41)$$

which clearly shows isolation of the real and imaginary parts so that the magnitude can be computed (see problems at end of chapter).

Earlier we discussed very briefly the idea that $T(s)$ and $g(t)$ (the impulse response) are Laplace transform pairs. Since it is usually easiest to obtain $T(s)$ by network analysis, we can get $g(t)$ by taking the inverse Laplace transform of $T(s)$. In almost all cases of practical interest, the inverse transforms needed are available from tables, although it is often necessary to adapt and/or combine entries, and to ignore countless entries that are of little or no practical value in signal processing work. Another point about using tables is that while the compilers of tables usually use the Laplace variable s (sometimes p), they do not feel obligated to maintain corresponding time/frequency dimensions, evidently thinking of variables in both domains as dimensionless. This can lead to some confusion in electrical engineering work. Some comments on this will follow the examples below.

EXAMPLE 1-6 Find the impulse response of the RC low-pass for Fig. 1-3, Example 1-1.

Here $T(s) = 1/(1+sCR) = (1/RC)/(s + 1/RC)$ so we look for this general form in tables and find something like $F(s) = 1/(s-a)$ paired with $f(t) = e^{at}$. Using this we get the impulse response as a decaying exponential:

$$g(t) = \frac{1}{RC} e^{-t/RC} \quad (1-42)$$

If we consider the dimensions of equation (1-42), we find that $g(t)$ has units of frequency due to the $1/RC$ multiplier, while we would expect it to have units of volts, or with more thought, whatever units the exciting impulse had. We can understand this appearance of frequency dimensions since we have transformed what we know to be a dimensionless $T(s)$. If we look at the inverse Laplace transform integral expression [inverse to equation (1-3)]:

$$f(t) = \frac{1}{2\pi j} \int_{a-j\infty}^{a+j\infty} F(s) e^{st} ds \quad (1-43)$$

we see that the ds term automatically gives us the frequency dimension that we found in equation (1-42). Likewise, the dt in equation (1-3) supplies an extra dimension of time. Accordingly, in cases where we do both a forward and an inverse transform, the dimensions will come out correctly, which we will see in later examples where step response rather than impulse response is desired.

Another point about impulse response is that the concept of an impulse in the analog case is difficult, having the interpretation of a Dirac δ -function which has infinite height and zero width for a net unit area. Possibly the safest approach to impulse response is as follows. First, treat $g(t)$ as a mathematical device [the inverse Laplace transform of our better-understood $T(s)$], and not so much as an observable laboratory waveform. Secondly, be aware that the general functional form (waveform if you prefer) of $g(t)$ is essentially correct even if the scaling is suspect. Finally, most or all of our difficulties will go away if we just consider a finite height, finite width (but relatively narrow) pulse rather than the idealized impulse.

EXAMPLE 1-7 Find the impulse response of the bandpass transfer function:

$$T(s) = \frac{As\omega_0}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2} \quad (1-44)$$

Typically tables will give two pairs that look "close".

$$\frac{1}{(s-a)^2 + b^2} \leftrightarrow \frac{1}{b} e^{at} \sin(bt) \quad (1-45)$$

$$\frac{s-a}{(s-a)^2 + b^2} \leftrightarrow e^{at} \cos(bt) \quad (1-46)$$

By combining these to cancel the constant term in the numerator (leaving the first power of s), one can show that the desired impulse response is:

$$g(t) = A\omega_0 e^{-\frac{\omega_0}{2Q}t} \left[\cos(\omega_0 \sqrt{1 - 1/4Q^2})t - \frac{1}{\sqrt{4Q^2 - 1}} \sin(\omega_0 \sqrt{1 - 1/4Q^2})t \right] \quad (1-47)$$

Again we see the confusion with the constant multiplier which has units of frequency, but it is clear that the waveshape, that of an exponentially decaying sinusoidal (as long as Q is positive) is correct. This is an example of the "ringing" of a filter, and will be discussed more in Chapter 3.

So far we have used the inverse Laplace transform to obtain $g(t)$ from $T(s)$, and have seen this to be of some utility, but also to have some problems. However it is also possible to find $v_{out}(t)$ for inputs other than an impulse. We can use

$$v_{out}(t) = L^{-1}[T(s) \cdot v_{in}(s)] \quad (1-48)$$

as long as we can find $V_{in}(s)$ from $v_{in}(t)$ and know how to invert the product $T(s) \cdot V_{in}(s)$.

EXAMPLE 1-8 Find the step response of the RC low-pass of Fig. 1-3, Example 1-1.

The problem is depicted in Fig. 1-9 where a step occurs at the input at $t=0$. We already know that $T(s) = 1/(1+sCR)$, and the Laplace transform of the unit step

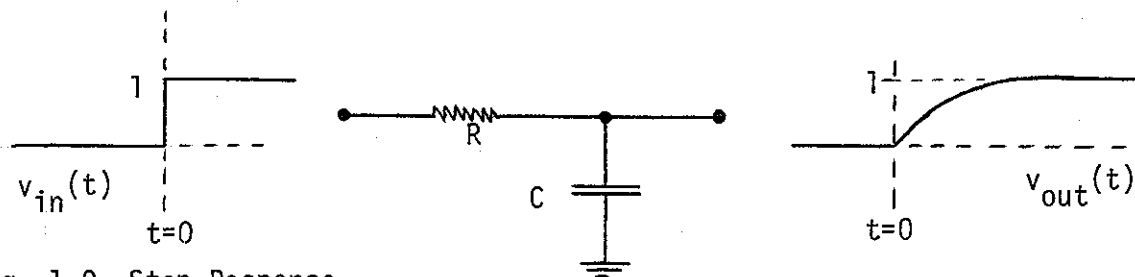


Fig. 1-9 Step Response

is just $1/s$. Thus the output is:

$$V_{out}(s) = V_{in}(s)T(s) = \frac{1}{s(1+sCR)} \quad (1-49)$$

We note that $V_{out}(s)$ in equation (1-49) has units of time, due to the forward transform from $v_{in}(t)$ to $V_{in}(s)$.

In order to invert $V_{out}(s)$, we need to expand to partial fractions form:

$$V_{out}(s) = \frac{-1}{s + 1/RC} + \frac{1}{s} \quad (1-50)$$

The two terms are now inverted separately:

$$v_{out}(t) = -e^{-t/RC} + 1 = (1 - e^{-t/RC}) \quad \text{for } t \geq 0 \quad (1-51)$$

which is the correct answer according to our knowledge of the charging RC circuit from physics. Note that the result has no time or frequency units. Instead it has whatever units we assigned to the step - voltage in this case. Note how nicely the scaling and unit problems went away when we took both the forward and the inverse transforms. Note finally that the Laplace method gave us the same result we would have gotten by solving the differential equation:

$$i(t) = dq(t)/dt = C dv(t)/dt = [v_{in}(t) - v_{out}(t)]/R \quad (1-52)$$

which the reader may wish to solve in the time domain so as to better appreciate the Laplace method.

1-7 OPERATIONAL AMPLIFIERS

Active filtering requires the use of some active device, usually an operational amplifier (or "op-amp" for short). The goal is to allow the realization of useful filters, having complex conjugate pole pairs; using only resistors, capacitors, and some active device (no inductors). In order to do this, the active device may serve to provide voltage amplification, to prevent loading of some circuit point (buffering), or an op-amp may serve directly. In an effort to understand op-amps, we will begin in this section with a study of the "ideal" op-amp, which is often a good and satisfactory model for the real thing. The effects of real op-amps will be discussed in Chapter 7.

Fig. 1-10 shows the conventional triangular symbol for an op-amp. It is usually considered a three-terminal device, with two inputs (non-inverting or v_+ and inverting or v_-), and one output v_{out} . In the figure we are actually showing two other terminals, those for the power supply voltages, which are usually ± 15 .

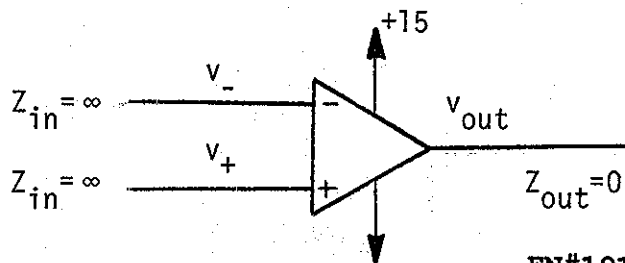


Fig. 1-10

The Ideal Op-Amp

In cases where we consider real op-amps, design circuits with them, or experiment with them in a lab, we of course have to deal with the need to supply them with appropriate power. However, it is also nearly a universal practice to leave them off of diagrams - unless they are something unusual.

Here, even in this case of an ideal op-amp, we are showing the power supply connections. [In short order we will start leaving them off.] The point is first to remind ourselves that we always need them in practice, but more importantly, they will help us realize at this early stage what the op-amp does and how it does it. Through the power supply lines, the op-amp has access to a source of energy that is external to the network. By way of its output, it can inject energy into the network in a controlled manner, as it senses network conditions by way of its two inputs. Thus we do not rely entirely on an input terminal from an external source to "excite" the network. The network can have a "self-exciting" aspect to it as well.

The input and outputs of the op-amp are considered ideal. For an output to be ideal, it is an ideal zero-output-impedance voltage source. It can't be "loaded down" by components attached to its output. It will "drive" whatever we attach. The op-amp output absolutely determines what the voltage at its output node is going to be. In sharp contrast, the inputs have nothing at all to say about what the voltages they are attached to are, except as they may be able to influence the output to alter network conditions. They are infinite-input-impedance inputs. They neither draw nor source current. You can attach them anywhere in the network and they will sense the voltage that is there, and it will be the same voltage that would have been there without them. They are perfect "probes."

This sets the stage for the most important of all the ideal op-amp properties. The op-amp is a differential amplifier and its output is accordingly given by a product of a differential input ($v_+ - v_-$) and some gain factor A . Thus:

$$v_{out} = A(v_+ - v_-) \quad (1-53)$$

For the ideal op-amp, A is taken to be infinite. It may be a surprise that anything practical can be accomplished by working with an idealized element with an assumed infinite gain. We will see below that it can, mainly in that whether or not A is infinite matters according to whether or not another parameter, ($v_+ - v_-$), is actually zero or just very very small. [In fact, it does not matter much if A is infinite, perhaps a mere 10^6 , or perhaps only 100. What matters more, as we will see in Chapter 7, is that in order to build stable op-amps, it is necessary to make A not just a constant, but a function of s , $A(s)$. Interestingly, we will see that this function of s is the first-order low-pass that has appeared in so many of our examples already in this chapter. Notice that the assumption that A is infinite and a constant automatically signifies that the op-amp has infinite bandwidth as well. The frequency response is infinite because there is no frequency dependence at all.]

We need to consider just how the ideal op-amp behaves, and we will begin by considering cases where the differential input voltage is specifically not equal to zero. Fig. 1-11 shows a number of such cases. In Fig. 1-11a we show how ordinary batteries might be used to bias the inputs for a differential value of +4.5. Now it is clear from equation (1-53) that if A is infinite, then v_{out} should certainly be infinite as well. Infinite output voltage is not one of the ideal op-amp assumptions however. We will assume that ideal op-amps, like real ones, stop trying when they clip against their power supply limits. Accordingly the output of the op-amp of Fig. 1-11a would be "clipped" at the positive supply of +15. [In case you are interested, a real op-amp would clip about two diode drops below the supply level, or at about ± 13.8 volts.] So if we make the differential input voltage non-zero, the op-amp just goes to one power supply level or the other, according to the polarity of the differential input.

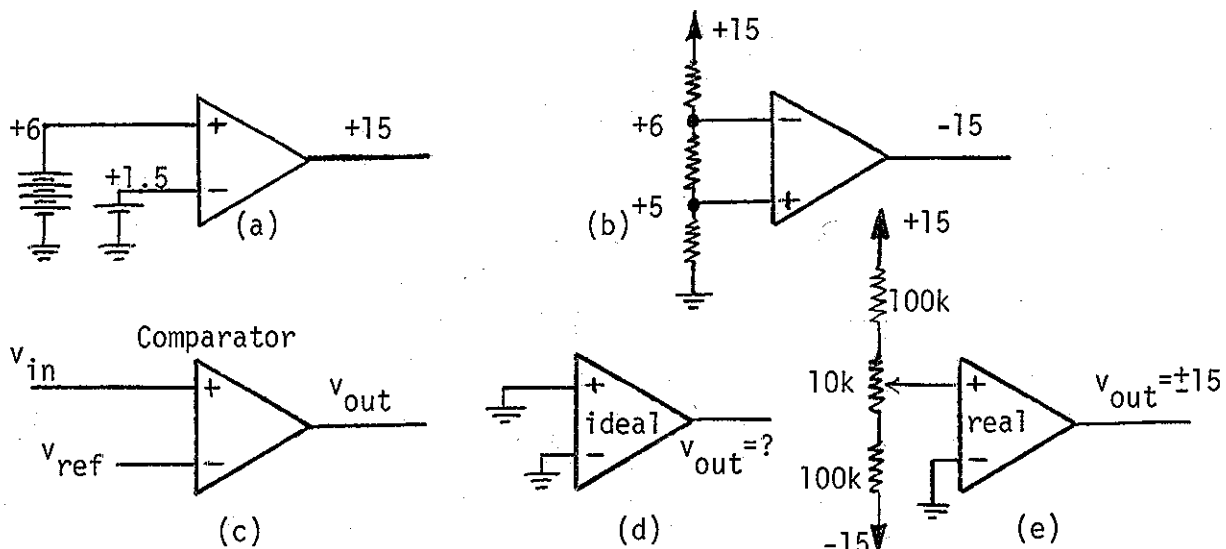


Fig. 1-11 Performance of Op-Amps

However we need not go to the trouble of bringing in batteries, but can just get reference voltages from a voltage divider as seen in Fig. 1-11b. Here the resistor values are not important, but only the ratio matters, since no current is drawn from the divider string by the inputs. Note that this biasing has v_+ more negative than v_- , so in this case the output is pinned at the negative supply.

These examples point out an important non-linear application of op-amps - that of the so-called comparator (Fig. 1-11c). The comparator can make a judgement about the input potential relative to a reference potential, and pin the output at \pm supply accordingly. When $v_{ref}=0$, the comparator is appropriately called a "zero crossing detector." Comparator applications are described here only for purposes of general information, and to illustrate how op-amps behave. We do not use comparators in linear circuits (not in active filters). Occasionally however we do accidentally get comparator response due to a circuit error. In testing op-amp circuitry, if an op-amp output is not supposed to be pinned at supply voltage, but is, it is usually not the case that the op-amp chip is faulty. Rather, as a simple check will test, usually the input voltages are separated and of a polarity that causes the op-amp output to pin. The particular op-amp is behaving correctly, as far as we know, but something feeding into it is wrong, and it must be corrected if we are trying to build a filter or other linear circuit.

We might now begin to wonder about how we are ever going to obtain a differential input voltage of zero. Fig. 1-11d shows one obvious approach, or we could just short the inputs together. Of course, this is just an experiment - we do not anticipate anything useful happening. With the ideal op-amp, since A is infinite we have a "zero times infinity" problem to worry about. However, if A is just "very large" we could suppose that $v_{out}=0$. If we changed Fig. 1-11d to a real op-amp however, there would be a voltage offset in the input stage (due to fabrication tolerances) and the output would pin at one supply or another. As a curiosity, we can also consider the circuit of Fig. 1-11e which uses a real op-amp. Is it possible to adjust v_+ by hand close enough to zero so that the output is not pinned at one of the supply rails? Perhaps for a moment or two, with a very steady hand on the pot control knob. It should be clear however that it is exceedingly tough to get the op-amp output to take on any voltage between the two supply rails - with what we have looked at so far.

What we have not done so far is to consider the possibility of using any sort of feedback from the output to an input - the devices we have looked at so far are what we call "open loop." It is through the use of negative feedback that we will achieve useful voltage values at the output, and a zero differential input. Our design rule with op-amps will then be extremely simple:

$$v_+ = v_-$$

(1-54)

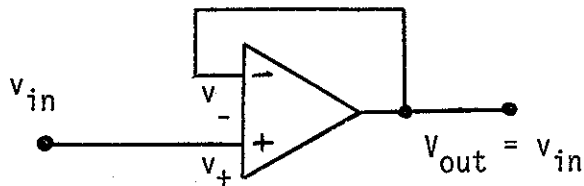


Fig. 1-12 The Unity-Gain Voltage-Follower of "buffer"

Fig. 1-12 shows an op-amp circuit called a unity-gain voltage-follower or "buffer" as it is often called. We will be concerned with this circuit in two ways. First, we will use it to demonstrate how negative feedback can result in a zero differential input. Secondly, the circuit itself is one of considerable utility in circuit work. While it provides no gain, it can be used to "buffer" a network point, as will be discussed more below.

To see how negative feedback works, let's begin by assuming that equation (1-54) is correct, and then examine what happens if the op-amp output tries to fluctuate. In particular, from Fig. 1-12 and equation (1-54) it is clear that the buffer should give:

$$v_{out} = v_{in} \quad (1-55)$$

Now, let's assume that v_{out} goes slightly positive above v_{in} . This will result in v_- going slightly positive with respect to v_+ , creating a negative differential input voltage, which will force v_{out} downward. Thus negative feedback tries to correct for the upward fluctuation. It is just as easy to show that a downward fluctuation of v_{out} will be corrected in a similar manner. The only possible value of v_{out} in this configuration is v_{in} .

Fig. 1-12, the follower, is unique in that the circuit has full or 100% negative feedback. Other circuits may have a different feedback path arrangement, but the principle that $v_- = v_+$ remains the same. As long as a negative feedback path is possible and working, ideal op-amp analysis begins with $v_- = v_+$. [In some real cases, negative feedback might fail because the output would be asked to supply a voltage outside its supply limits, or the input might move too fast for the output to keep up exactly.]

It is somewhat important to understand the situation at the differential input from the proper point of view. It is true that v_- is forced to take on the same potential as v_+ , but it is not true that the inputs themselves "force" this condition. Indeed, the inputs by themselves have no "influence" on the voltage points to which they are connected - except as they are able to influence the output to help out. It is always the case that the output must do the work, to take on whatever value is necessary so that v_- becomes the same as v_+ . However, because it is the usual starting point to assume that $v_- = v_+$, it is somewhat natural to have the impression that a current flows because of the $v_- = v_+$ condition, and that this current forces itself through some impedance and forces the output to take on some value. A more correct point of view is that the output tries some value, and then adjusts it until $v_- = v_+$ through the negative feedback process.

It should be realized that while the follower provides no gain, it does provide a buffering action of great utility. [Indeed we often don't even want gain for a certain application.] If we have a point in a circuit which has a voltage that we would like to extract and use elsewhere, we need to be concerned with whether or not the attached circuitry will "load" the original point and change the original voltage significantly. If so, we can use a buffer to offer a "refreshed" version of the original voltage. This occurs because the buffer has infinite input impedance and zero output impedance. Buffers have important uses in active filters and in analog circuitry in general.

Fig. 1-13 shows a second circuit, the non-inverting amplifier, which is like a buffer with added gain as well. In analyzing this circuit, what we need is to find

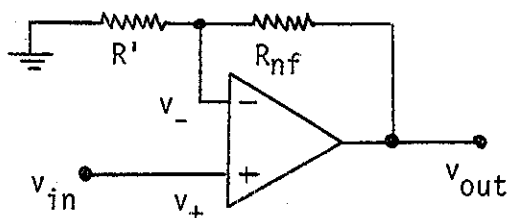


Fig. 1-13 Non-Inverting
Finite-Gain
Voltage Amplifier

the relationship between v_{out} and v_{in} , which is the same as finding the transfer function $T(s)$ of the circuit. Here however since there are no capacitors or inductors involved, and since the op-amp is ideal, the transfer function is not a function of s , and it is convenient to just consider it a constant K . The procedure in finding it is similar to that of finding a transfer function, none the less.

Analysis of Fig. 1-13 is a matter of realizing that a negative feedback path is available since v_{out} is fed back to v_- by a simple voltage divider:

$$v_- = v_{out} \frac{R'}{R' + R_{nf}} \quad (1-56)$$

but since $v_- = v_+ = v_{in}$ it follows that:

$$v_{out}/v_{in} = K = 1 + R_{nf}/R' \quad (1-57)$$

The circuit is similar to the follower in that it has infinite input impedance and zero output impedance, but this one has a gain of K that is given by 1 plus the ratio of resistors as shown. The circuit is a finite-gain non-inverting voltage amplifier and is also sometimes called a VCVS (for Voltage-Controlled Voltage Source.)

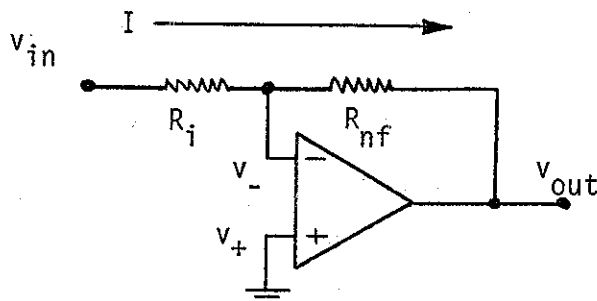


Fig. 1-14 Inverting
Finite-Gain
Voltage Amplifier

Fig. 1-14 shows a third circuit, the inverting amplifier. Again we want to find the relationship between v_{out} and v_{in} , starting with the principle that a negative feedback path is available and that $v_- = v_+$. However in this case v_+ is grounded so $v_- = v_+ = 0$, and in such a case, the v_- terminal is often called a "virtual ground." This means that the $(-)$ input is held at ground potential. However, no current actually flows to ground at v_- . In fact, since for the ideal op-amp, no current flows into the $(-)$ input at all, only one current I flows in through R_i and out through R_{nf} as shown. This suggests an analysis based on this single current. Since $v_- = 0$, it is clear that:

$$I = v_{in}/R \quad (1-58)$$

and this current flowing out through R_{nf} , from positive to negative, means that:

$$v_{out} = 0 - I \cdot R_{nf} = -v_{in} \frac{R_{nf}}{R'} \quad (1-59)$$

so we have:

$$v_{out}/v_{in} = K_i = -R_{nf}/R' \quad (1-60)$$

Thus Fig. 1-14 has an inverting gain equal to the ratio of resistors shown. In contrast to the follower and the non-inverter however, the input impedance here is not infinite, but rather equal to R_i , as can be seen from equation (1-58) since the

input impedance would be defined as the ratio of the input voltage to the input current. The difference here is that we are not able to take advantage of a "bare" op-amp input terminal.

Above we mentioned the need for the correct point of view with regard to the way negative feedback works, and the analysis of Fig. 1-14, while correct, may imply the wrong point of view in that the current I seems to be a cause rather than an effect. An alternative analysis may serve to provide a different point of view and also to illustrate that there are usually several valid analysis procedures. We assume that v_{in} is from a zero impedance source, and v_{out} is also a zero impedance source since it is the output of our op-amp. Accordingly we have two resistors in series with the junction "floating" according to (see problems at end of chapter):

$$v_- = \frac{v_{out} \cdot R_i + v_{in} \cdot R_{nf}}{R_i + R_{nf}} \quad (1-61)$$

Now, v_- must be equal to zero, so equation (1-61) results in equation (1-60). Here we more clearly see that v_{out} takes on whatever values is necessary so that v_- becomes zero.

A couple of additional circuits are shown in Fig. 1-15, an inverting summer, and a finite-gain differential amplifier. It is left to the reader to apply the idea that $v_- = v_+$ and to show for the inverting summer that:

$$v_{out} = -v_1 \frac{R_1}{R_{nf}} - v_2 \frac{R_2}{R_{nf}} \quad (1-62)$$

and for the differential amplifier that:

$$v_{out} = v_2 - v_1 \quad (1-63)$$

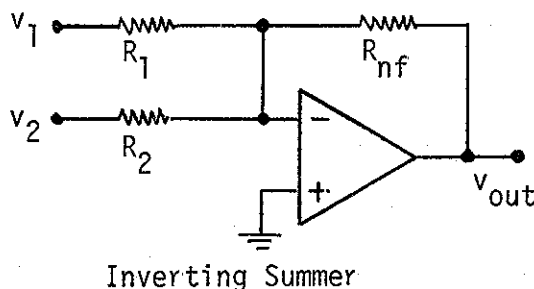
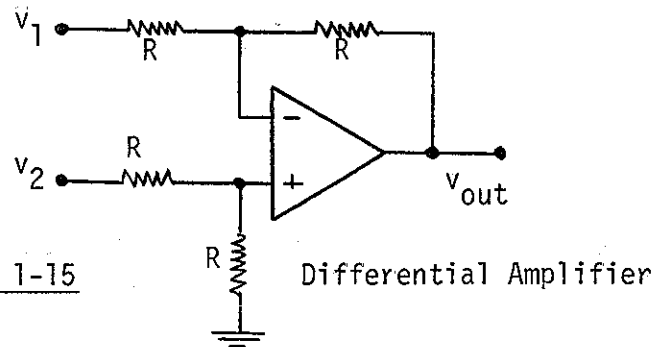


Fig. 1-15



The above illustrate common and important op-amp circuits, but there are many others, all of which can be solved in the ideal case by starting with $v_- = v_+$. It is important to understand these circuits, as active filters may use them directly, or they may be variations on them. For example, none of the results requires that what we have shown as resistors can not be some other impedance. We could substitute for a resistor the impedance $(1/sC)$ of a capacitor C if we have a capacitor there instead. Or, the impedance might be the combined impedance of two or more components in a particular branch. The problems at the end of the chapter, and the methods and circuits in later chapters will expand on these ideas.

CHAPTER 2

ACTIVE FILTER EXAMPLES LEADING TO COMPLEX CONJUGATE POLES

- 2-1 Introduction
- 2-2 R-L-C Series Circuit
- 2-3 Sallen-Key: An Active Approach
to Complex Poles
- 2-4 Multiple-Feedback Infinite-Gain:
An Op-Amp Circuit for Bandpass

Nearly all filters of practical interest will have most or all their poles in complex conjugate positions in the s-plane, as will be seen in Chapter 3. For stability, these poles will need to have a negative real part, and because we are working with real-valued components, complex poles must be in conjugate pairs.* In order to realize complex conjugate poles with only passive elements, it is necessary to employ both capacitors and inductors. Inductors in the RF range of frequencies are often practical, but at audio frequencies and below, they are usually too large and too heavy. Consequently, it is the goal of active filtering to achieve complex conjugate pole pairs while using only resistors, capacitors, and some active device (usually an operational amplifier). In this chapter, we will see two examples where op-amps are used: the Sallen-Key low-pass, and the Multiple-Feedback Infinite-Gain bandpass. First however, we will look at a passive RLC series circuit, where the conjugate poles are obtained with an inductor.

2-2 THE R-L-C SERIES CIRCUIT

Fig. 2-1 shows the classic R-L-C series circuit in three possible arrangements of the component order which permits us to conveniently output either the voltage across the capacitor (a), the resistor (b), or the inductor (c). It is easy to see that the ordering makes no difference at the input, and a current:

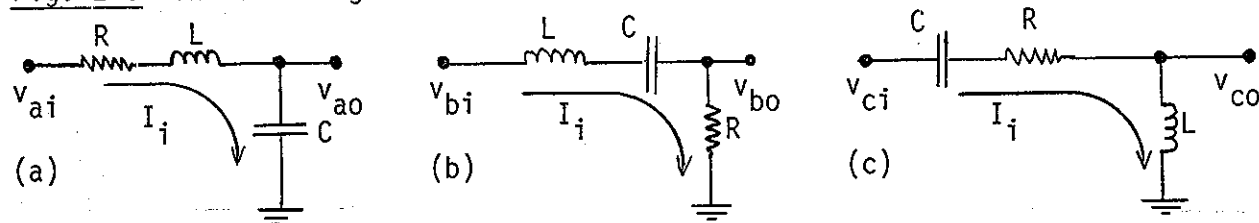
$$I_i = v_i / (R + sL + 1/sC) \quad (2-1)$$

flows in all three cases. Further, this same current flows through all three of the components, and thus generates the output voltage:

$$v_o = I_i \cdot Z_o \quad (2-2)$$

where Z_o is the impedance in the output leg for the particular case. Accordingly we can write down the transfer function for case (a) using the voltage-divider

Fig. 2-1 Three Arrangements of the R-L-C Series Circuit



*Accordingly, if a pole occurs at $\sigma + j\omega$ in the s-plane, σ must be negative for stability, and a second pole must occur at $\sigma - j\omega$. It is our notational convention to write σ with the knowledge that usually σ is a negative number, and not to write $-\sigma$ with the knowledge that σ is usually positive. Note that although a complex conjugate pole pair results in a second-order factor for the denominator of $T(s)$, the poles are first-order. In order to have a second-order pole, two poles must occur in the exact same position in the s-plane. The order of $T(s)$ is equal to the number of poles. For example, a fifth-order system might have poles at $\sigma_1 + j\omega_1$, $\sigma_2 + j\omega_2$, $\sigma_1 - j\omega_1$, $\sigma_2 - j\omega_2$, and at σ_3 . Note that odd ordered transfer functions must have at least one real pole. Multiple order poles occur in the following examples. A fourth order $T(s)$ could have two poles at $\sigma + j\omega$, and two at $\sigma - j\omega$. A fifth order $T(s)$ could have poles at $\sigma + j\omega$, $\sigma - j\omega$, and a third-order pole at σ_2 .

concept as:

$$T_a(s) = v_{ao}/v_{ai} = \frac{1/sC}{R + sL + 1/sC} = \frac{1/LC}{s^2 + s\frac{R}{L} + \frac{1}{LC}} \quad (2-3)$$

and similarly for cases (b) and (c):

$$T_b(s) = v_{bo}/v_{bi} = \frac{s\frac{R}{L}}{s^2 + s\frac{R}{L} + \frac{1}{LC}} \quad (2-4)$$

$$T_c(s) = v_{co}/v_{ci} = \frac{s^2}{s^2 + s\frac{R}{L} + \frac{1}{LC}} \quad (2-5)$$

We note immediately that all three transfer functions have the same denominator $s^2 + s(R/L) + 1/LC$, with corresponding poles:

$$s_{p1,p2} = \frac{-R}{2L} \pm \frac{1}{2} \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}} \quad (2-6)$$

which has non-zero imaginary part if $4/LC > R^2/L^2$, thus if $R < 2\sqrt{L/C}$. Thus we have imaginary poles if R is small enough.*

Fig. 2-2 shows how the poles occur in the s -plane as a function of R . Complex poles occur for values of R between 0 and $2\sqrt{L/C}$. We can reverse the terms in the square root of equation (2-6), thereby bringing a j outside as:

$$s_{p1,p2} = \frac{-R}{2L} \pm \frac{j}{2} \sqrt{\frac{4}{LC} - \frac{R^2}{L^2}} \quad (2-7)$$

[Note that equations (2-6) and (2-7) are identical. Equation (2-7) just shows the imaginary part better, and is more suitable for our purposes here. In the event that R^2/L^2 is actually larger than $4/LC$, equation (2-7) would give real poles once again.] The magnitude of the complex poles is, using equation (2-7):

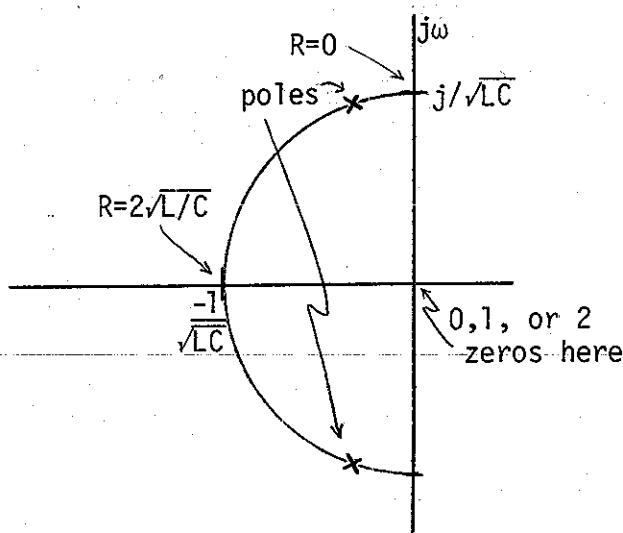


Fig. 2-2 Pole Positions of R-L-C Circuit for Various Values of R

*In fact, this relates to the classical study of the R-L-C series circuit which is known to have a higher "Q" ("quality factor"), or higher selectivity if R is kept small. Later we will see that a quantitative notion of "Q" here yields $Q = (1/R)\sqrt{L/C}$ in which case complex poles occur for $Q > 1/2$.

$$|s_{p1}| = |s_{p2}| = 1/\sqrt{LC}$$

(2-8)

so complex poles lie on a circle of radius $1/\sqrt{LC}$. When $R=0$, the poles are pure imaginary at $\pm j/\sqrt{LC}$, corresponding to an oscillator (an undamped LC circuit - which is not possible in practice since the inductor has at least some resistance). The poles move around the circle, backward into the left half plane, as R increases from zero. The poles come together forming a second-order real pole at $-1/\sqrt{LC}$ when R reaches $2\sqrt{L/C}$. If R is larger than this value, the poles remain real, splitting from their $-1/\sqrt{LC}$ position with one moving toward $s=0$ while the other moves toward $s=\infty$ as R goes to ∞ . We note that the poles can not move into the right half plane, since this would require a negative value of R (or negative values of other components). This absolute stability of a network is characteristic of passive networks, but as we shall see, not of active networks in all cases.

In addition to understanding that the R-L-C circuit is capable of producing complex poles, we see here that different transfer functions are also possible [equations (2-3, 2-4, and 2-5)] which differ essentially in the power of s that occurs in the numerator. This corresponds to there being no zeros, one zero, or a second-order zero at $s=0$ in the s -plane (Fig. 2-2). We can consider what this means for the frequency response function by restricting s to $j\omega$, and then look at the limits of $|T(s)|$ for different values of ω . Accordingly we see for $T_a(s)$ that we have a low-pass function, starting at 1 for $\omega=0$, and rolling off as $1/\omega^2$ as ω gets large. $T_b(s)$ is a bandpass function which starts at zero for $\omega=0$, rolls up as ω , and finally rolls off as $1/\omega$ at high frequencies. $T_c(s)$ is a high-pass function, starting at zero for $\omega=0$, rolling up as ω^2 , and becoming flat at a value of 1 for high frequencies. These responses and limits can also be understood in terms of the way in which the individual R , L , and C components behave at different frequencies. In addition, while we have looked at limits of low and high frequency, the behavior in the middle depends greatly on where the poles are. As the poles move closer and closer to the $j\omega$ -axis, the responses of all three transfer functions tend to peak in the center, corresponding to our idea that the frequency response is inversely proportional to the distances to the poles. Selecting the correct pole positions for a desired response is a major part of what we need to consider in Chapter 3. Fig. 2-3 shows sketches of the three frequency response functions discussed above, for the case where the poles are approximately as in Fig. 2-2.

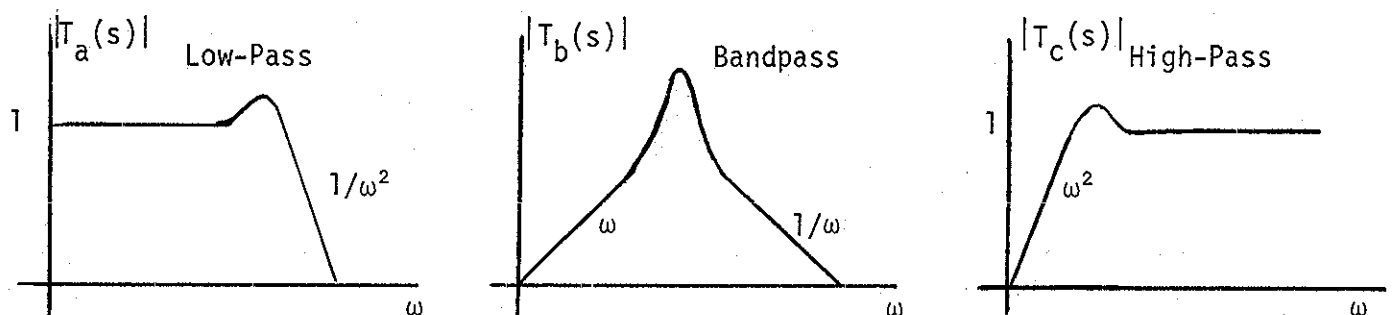


Fig. 2-3 Low-Pass, Bandpass, and High-Pass Functions Available from the R-L-C Series Circuit.

The R-L-C circuit has two poles and behaves like other second-order systems, including mechanical systems such as the simple spring-mass arrangement of Fig. 2-4. Although we do not show it, we know that frictional forces will always be present to provide mechanical damping (the resistive component) in a practical case. We can become interested in mechanical systems in direct analogy with electrical ones. Here however we want to point out that we can take advantage of the good mechanical intuition that most of us already have. Probably we have played with a system similar to that of Fig. 2-4 (possibly it was something more like a soda can suspended

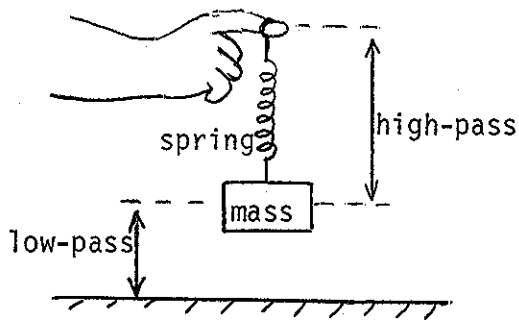


Fig. 2-4 A Simple Spring-Mass
Second-Order System
(Where is Bandpass?)

below a string of rubber bands!). Moreover, we can ask certain "What happens if . . . ?" questions, and we can usually think it out based on experience and intuition. Very likely we could not intuitively have answered the corresponding electrical case.

The main point in looking at this mechanical analog is to understand the role played by resonance in a second-order system. In order to consider this properly, we should identify low-pass, bandpass, and high-pass modes in the mechanical system, if they are present. It is clear that as the finger moves up and down, that the spring and mass respond.

Let's consider low frequencies first. As the finger moves slowly, the mass moves up and down with it, with the spring remaining pretty much at its original (stationary) length. Now if the hand moves rapidly, we recognize that the inertia of the mass will cause it to stand relatively still while the spring stretches and contracts to accommodate the finger's motion. Thus we see that the motion of the mass, relative to a fixed reference, is low-pass in nature. Correspondingly, the elongation of the spring (about its stationary elongation) is high-pass. Is there a bandpass function here? What is it that happens special at the middle range of frequencies?

Intuitively or from experience we know that the system can "go crazy" over some range of middle frequencies. We know that there is a range of middle frequencies for which we will start to fear for the safety of our teeth or for a nearby window, because the mass has a good deal of something potentially destructive. What that something is that there is an alarming amount of is energy - the system is trading off between potential (spring elongation) and kinetic (mass velocity) energy, and the total is large. It is probably most convenient to think however in terms of the peak velocity, the peaks in kinetic energy, as representing the bandpass function. We recognize this peak region of response as being resonance - the region where the forcing frequency is close to the natural frequency of the system - and we tend to associate resonance with bandpass filters of good selectivity.

Having set up the discussion, we specifically want to point out that resonance, while associated with bandpass, also occurs in low-pass and high-pass responses, as suggested in Fig. 2-3. Note that for low-pass at low frequencies, the amplitude of the motion of the mass is the same as that of the finger - essentially what we call unity gain. However, unless the spring is very stiff we also find that there is a region where the mass starts moving through a somewhat greater amplitude range than the driving finger. Likewise in the high-pass case, at high frequency the mass is standing still with the spring elongation accommodating the finger's motion (again unity gain). For some lower frequencies however, we do find the elongation exceeding the finger's motion as the mass moves a significant amount (and with the proper phase). How we view peaking in any response should be regarded in the way resonance manifests itself. If the system is highly resonant (high "Q" or poles close to imaginary axis), peaking will occur in low-pass, bandpass, and high-pass responses. In fact, all three responses may look very much alike in the middle range.

In addition to considering resonance, we have suggested above that some thoughts with regard to mechanical analogs are useful to take advantage of our mechanical intuition. For example, if we are asked what the impulse response of a high-pass

filter is, we might not have much of an idea from an electrical perspective, and would have to work out the mathematics. However, we do know what happens if we apply an impulse to the mechanical system. If we simply move the finger quickly and restore it quickly to its original position, what does the system do? It is left to the reader to think this out. Once you know what the system does, we can then see how this looks in the response modes we have identified.

2-3 SALLEN-KEY: AN ACTIVE APPROACH TO COMPLEX POLES

One notion of a way to achieve a higher order low-pass network might be to cascade two first-order low-pass filters of the type examined in Chapter 1, in an arrangement as suggested by Fig. 2-5. However, it is not difficult to show (see problems at end of chapter) that complex conjugate poles are not possible here, but only two real poles are achieved for any values of the components.

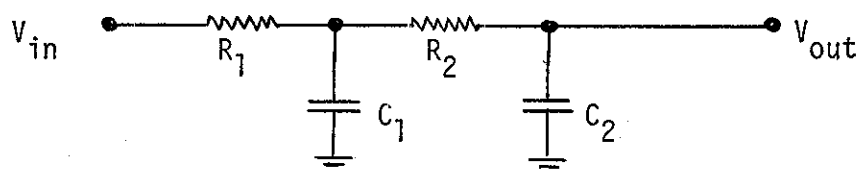


Fig. 2-5
Cascade of Two
Passive Low-Pass
Sections

By a basically unmotivated step, let's consider what happens if we add a voltage amplifier of gain K to the network output, and feed the amplified output back to a previously grounded resistor, as shown in Fig. 2-6a. The amplifier K is usually formed from a non-inverting op-amp stage (Fig. 2-6c) which was studied in Chapter 1.

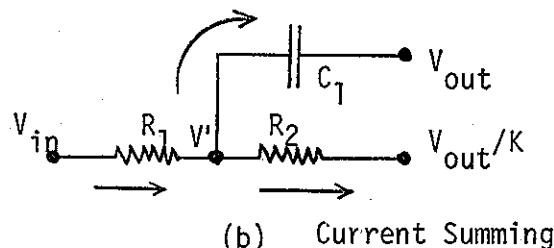
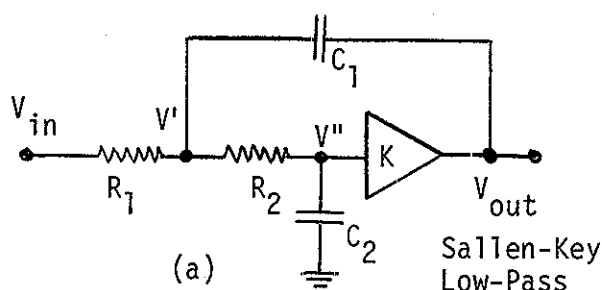
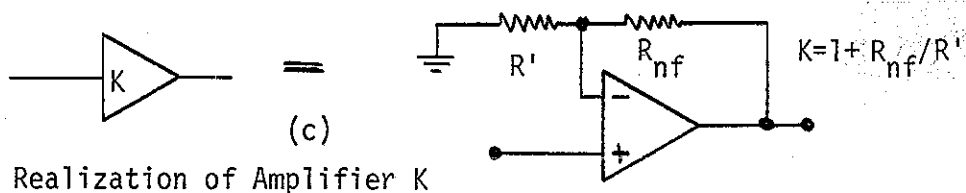


Fig. 2-6
Sallen-Key
Low-Pass



Our initial goal is to determine the transfer function of the network using network analysis. We know V_{in} , but we do not know V_{out} , or the two intermediate voltages V' or V'' . We might look at this as three unknowns and look for three equations. However, it is often possible and more productive to just work your way through the network, one item at a time. Useful and practical networks usually have an uncomplicated structure that makes this the best approach. Here we will work item by item as an example.

The first thing to do is to note that $V_{out} = K \cdot V''$, and since we do not want V'' for $T(s)$, we can replace it with V_{out}/K . This is of course one of the three equations we would have needed in a formal approach. The idea here is that we get rid of the equation by just considering this a change of notation for the node V'' - something too obvious to make a big deal about.

The second thing we do is to relate V' to V_{out}/K (formerly called V''). If we

Look carefully, we recognize our familiar first-order low-pass here, and can just write:

$$\frac{V_{out}}{K} = \frac{V'}{1 + sC_2R_2} \quad (2-9)$$

One might initially object to applying the first-order low-pass in this case since V' is not a zero-impedance source as it was considered to be in Chapter 1. However, the analysis depends only on the voltage divider ratio, and it works just as well here.*

We have now related V_{out} all the way back to V' , which is our remaining unknown. We obtain the final relationship we need by summing currents at the V' node:

$$\frac{V_{in} - V'}{R_1} = \frac{V' - V_{out}}{1/sC} + \frac{V' - V_{out}/K}{R_2} \quad (2-10)$$

in a manner illustrated in Fig. 2-6b. **

Solving equations (2-9) and (2-10), we arrive at the transfer function:

$$T(s) = \frac{K/R_1R_2C_1C_2}{s^2 + s\left[\frac{(1-K)}{R_2C_2} + \frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right] + \frac{1}{R_1R_2C_1C_2}} \quad (2-11)$$

Equation (2-11) has 5 variable parameters: R_1, R_2, C_1, C_2 , and K . Yet we are concerned mainly with setting a pair of complex poles in a desired position, which should require only two parameters (the real and imaginary parts). Thus we can choose three of the

* V' is loaded by the R_2 - C_2 branch in the sense that if it were not there, V' would surely be a different voltage, but it is the loaded version of V' that we are relating to V_{out}/K here, not the unloaded one.

** Here what we are doing is essentially Kirchhoff's current law which says that currents flowing into a node must sum to zero (conservation of current). However rather than show all currents flowing into a node, it is often more comfortable to have the sum of the currents into the node equal to the sum of the currents out of the node, which is mathematically equivalent. There are two advantages to the second approach. First, having all currents into a node is counterintuitive, while having at least one current leaving seems to make things more plausible. Second, if currents can be directioned mainly left to right and top to bottom, there is less chance of confusion. Consider for example the current through R_2 in Fig. 2-6a. Clearly this current has a magnitude equal to $|V' - V''|/R_2$. If we were summing currents to zero into nodes, then this current would be $(V'' - V')/R_2$ at the V' node, and $(V' - V'')/R_2$ at the V'' node, opposite in sign. On the other hand, if this current flows from left to right, then it is $(V' - V'')/R_2$ and is subtracted from the V' node and added to the V'' node, exactly the way it looks. Because this is more consistent with intuitive understanding, there is less chance of setup errors.

Another point with regard to current summing is that we do not sum currents at a node which is a voltage source. Rather we sum currents at "floating" nodes as an essential step in determining the voltage at those nodes. The voltage at a node driven by a voltage source is determined by whatever controls the voltage source. This node voltage is independent of whatever currents are flowing to or from this node through impedances from other voltage sources. The voltage source by definition can source or sink any current necessary to conserve total current. For example, in Fig. 2-6a, V_{out} is a voltage source node. It is controlled by the voltage V'' , but not at all by the current through C_1 .

five parameters at our convenience. Recognizing that in many cases it is difficult to obtain a good variety of capacitance values, we will choose $C_1 = C_2 = C$, where C is a convenient value of capacitance. Thus we have used two of our three free choices. Our third choice will be utilized by setting $R_1 = R_2$. We will write this as $R_1 = R_2 = R$, as we did for the capacitances, but there is a difference here. In specifying R , we mean only to say that the common value of R_1 and R_2 is being called R . We do not get to choose R freely, but this is usually not a problem. With these choices, $T(s)$ becomes:

$$T(s) = \frac{K/R^2 C^2}{s^2 + \frac{s}{RC} (3-K) + 1/R^2 C^2} \quad (2-12)$$

which has poles at:

$$s_{p1,p2} = \frac{1}{RC} \left[\frac{-(3-K)}{2} \pm \frac{1}{2} \sqrt{(3-K)^2 - 4} \right] \quad (2-13)$$

These poles are complex when $(3-K)^2 < 4$, and occur at:

$$s_{p1,p2} = \frac{1}{RC} \left[\frac{-(3-K)}{2} \pm \frac{j}{2} \sqrt{4 - (3-K)^2} \right] \quad (2-14)$$

which lie on a circle of radius $1/RC$. It is easy to show (see problems at end of chapter) that stable complex poles occur for values of K between $+1$ and $+3$. We now have complex poles without inductors.

Fig. 2-7 corresponds to Fig. 2-2 for the R-L-C circuit, except here it is possible for the poles to move to the right half-plane, and stability is not automatic. Equation (2-12) represents a low-pass filter, and note that we can control the angle of the poles by using different values of K . In Chapter 3 we will find out how to select K and the RC time constant for achieving desired frequency response curves. The circuit of Fig. 2-6a is one of several useful active filter networks described by Sallen and Key in 1955 [1] and is known as the Sallen-Key low-pass, or as a positive gain VCVS realization.

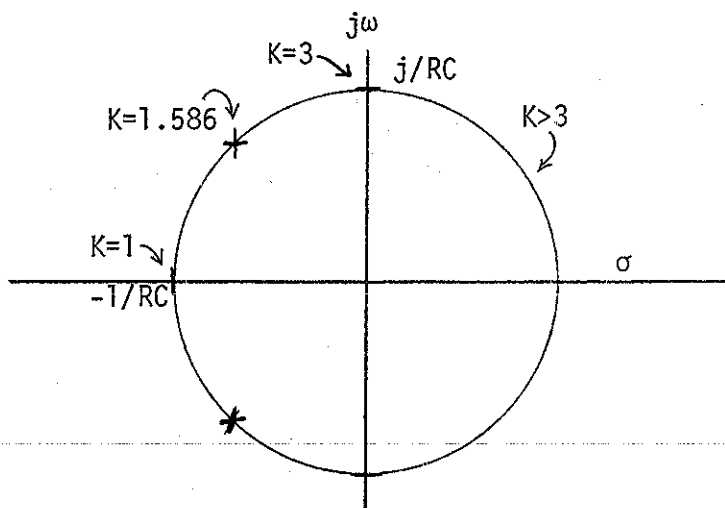


Fig. 2-7 Pole Positions as Function of K for Sallen-Key Low-Pass. Stable complex poles occur for K between $+1$ and $+3$. For K greater than 3 , the system is unstable. For K less than 1 , the system is stable and poles are real.

2-4 MULTIPLE-FEEDBACK INFINITE-GAIN: AN OP-AMP CIRCUIT FOR BANDPASS

This chapter is an introduction to active filtering so we feel compelled to give more than one example. The second example is a Multiple-Feedback Infinite-Gain (MFIG) bandpass filter, which we choose for several reasons. First, it shows an analysis using the op-amp directly (the infinite gain part of the name) and not used for a finite gain amplifier as in Sallen-Key. Secondly, it is a bandpass rather

than another low-pass example. Thirdly, the MFIG bandpass is the basis for numerous other filters which will be described in Chapter 5.

This MFIG structure and the Sallen-Key structure are examples of what we call "configurations." The idea is that we may have in mind a basic type of filter (for example, low-pass or bandpass or high-pass) and may even have decided on an exact "characteristic" for it [characteristics are the subject of Chapter 3]. We then need to choose a configuration that realizes the function we need. Sallen-Key is one configuration, for which we have looked at low-pass, and MFIG is another configuration for which we are about to look at bandpass. However, there are Sallen-Key bandpass filters and MFIG low-pass, and so on. We will see many more examples in later chapters.

Fig. 2-8a shows the configuration for the MFIG bandpass. All active filter analysis procedures should begin by considering what sort of active device is being used.* Here it is the op-amp itself, and for ideal analysis, we begin with $V_- = V_+ = 0$. The second step in the analysis should be to try to recognize something familiar as a sub-network within the overall network. In this case, it is the inverting structure that is detailed in Fig. 2-8b. This we will treat by generalizing the inverting amplifier of Fig. 1-14 of Chapter 1. We can write the transfer function $T'(s) = V_{out}(s)/V'(s)$ as:

$$T'(s) = -\frac{R_2}{1/sC} = -sCR_2 \quad (2-15)$$

Because it multiplies by s , the circuit of Fig. 2-8b is called a differentiator. It is well to become familiar with this and a fair number of other such simple sub-networks as they will greatly simplify and speed up analysis.

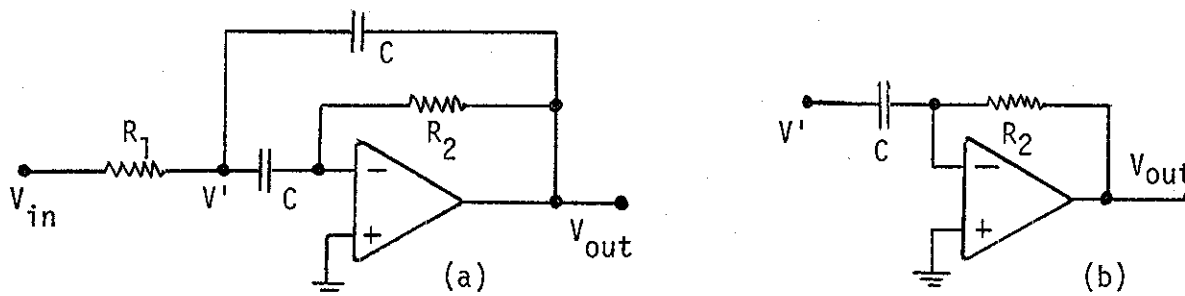


Fig. 2-8 The Multiple-Feedback Infinite-Gain Bandpass

Having now related V' to V_{out} by equation (2-15), we have only one of V' or V_{out} as a remaining unknown. We thus come down to a third step of summing currents at the V' node:

$$\frac{V_{in} - V'}{R_1} = \frac{V' - V_{out}}{1/sC} + \frac{V' - 0}{1/sC} \quad (2-16)$$

Solving equations (2-15) and (2-16) we arrive at the transfer function:

$$T(s) = \frac{-s/R_1C}{s^2 + \frac{2}{R_2C}s + \frac{1}{R_1R_2C^2}} \quad (2-17)$$

which is clearly a bandpass response since there is an s in the numerator.

* Here we specifically want the reader to understand the analysis steps that are typical. First we look at and understand the active element that is being used. Secondly, we look for some familiar sub-network within the network. The third step is almost always to sum currents at an unknown node (usually the only remaining unknown node). After following this example, the reader may find it useful to go back and identify the same steps in the Sallen-Key low-pass.

As with the R-L-C circuit and the Sallen-Key, equation (2-17) is capable of having complex conjugate poles. The poles are complex as long as $R_2 > R_1$ and are at:

$$s_{p_1, p_2} = \frac{1}{R_2 C} \left[-1 \pm j \sqrt{R_2/R_1 - 1} \right] \quad (2-18)$$

which lie on a circle of radius $1/\sqrt{R_1 R_2} C$. Note that these poles are always stable since the real part is always negative.

In Chapter 3 we shall study the general second-order section, but we want to introduce some ideas here by way of commenting on some common findings from this chapter. First we observe that we have put transfer functions in a form where the leading term in the denominator is s^2 [see equations (2-3), (2-12), and (2-17)]. In each case, we found complex poles on a circle, and if we check, we find that the radius of this circle was always the square root of the constant term (last term) in the denominator. Accordingly, if we denote the pole radius as ω_0 , then we may want to write the last term as ω_0^2 .

Pole positions would be exactly known if we knew the angle of the poles in addition to knowing the radius of the pole pair. Equivalently we need to know any two of the real part, the imaginary part, or the magnitude (radius) of the poles. For several reasons we will be looking at the real part of the poles and will be able to relate this to a notion of "damping."

First, the real part is immediately available as $-1/2$ times the coefficient of the s term in the denominator (using the quadratic formula and assuming the leading term in the denominator as s^2). Secondly, the real part tells us about the stability. Thirdly, the real part leads us to a notion of damping that we will find useful. [In fact, usually we will find the exact values for resistors and capacitors by specifying the pole radius and the damping of the poles, and then using appropriate "design equations"]. Note that the real part in itself is not totally significant - we must relate it to the pole radius. For example, a pole with real part equal to -100 Hz is "close" to the $j\omega$ -axis if the radius is $10,000$ Hz, but is not "close" if the radius is 110 Hz. By the same token, a real part of -100 Hz relative to a radius of 1000 Hz has the same fundamental placement in the s -plane as one with -500 Hz real part and 5000 Hz radius.

It is convenient to think in terms of a standard form for the denominator of a second-order section as:

$$D(s) = s^2 + D\omega_0 s + \omega_0^2 \quad (2-19)$$

where D is a term we will call damping, which has complex poles at:

$$s_{p_1, p_2} = -D\omega_0/2 \pm (j\omega_0/2)\sqrt{4 - D^2} \quad (2-20)$$

The real part is $\sigma_p = -D\omega_0/2$, so the damping is:

$$D = -2\sigma_p/\omega_0 \quad (2-21)$$

Notice that D , unlike the real part σ_p itself, is "normalized" to the radius ω_0 . All poles on the same radius out from $s=0$ in the s -plane have the same damping.

The parameters of pole radius ω_0 and of damping D permit us a link between the transfer function of a configuration on the one hand, and the set of poles and zeros we want to realize on the other hand. The connection is completed when we are able to write down a set of "design equations." These design equations allow us to compute actual resistor and capacitor values for the response we need in a given application.

Let's obtain the design equations for the transfer function of the MFIG bandpass

of equation (2-17). We need to have the denominator of equation (2-17) in the standard form of equation (2-19), from which we see that:

$$\omega_0 = 1/C\sqrt{R_1 R_2} \quad (2-22)$$

It must also be true that the middle terms agree, or:

$$D\omega_0 = 2/R_2 C \quad (2-23)$$

so that:

$$D = 2/\omega_0 R_2 C = 2\sqrt{R_1/R_2} \quad (2-24)$$

If we had ω_0 and D specified for our application, we could choose a convenient value of C and then solve equations (2-22) and (2-24) for R_1 and R_2 , completing the design. Equations (2-22) and (2-24) are themselves considered design equations, or it may be useful to solve them for R_1 and R_2 in terms of D and ω_0 if a more "cookbook" approach is desired.

Since this particular network is bandpass, it is often the case that the "Q" of the bandpass rather than the damping D is used. The relationship is simple:

$$Q = 1/D = (1/2)\sqrt{R_2/R_1} \quad (2-25)$$

In Chapter 3 we will show that for the bandpass, it is also true that:

$$Q = \omega_0/(\omega_u - \omega_l) \quad (2-26)$$

where ω_u and ω_l are the upper and lower -3db frequencies - the frequencies on each side of the bandpass response that are 3db down from the peak. Thus we have the usual (classic) definition of Q as the center frequency divided by the 3db bandwidth. The characterization of the bandpass response will be covered in more detail in Chapter 3.

ASP 2-10

ENDNOTES:

In quite a few places, the text refers to a "problem at the end of the chapter." Well, they are not here! These were never typed up. However, almost without exception the intended problems are highly evident from the text itself. We have left these references in because awkward blank areas would have resulted by whitening them out. More to the point, they are good indicators of places where the reader is expected to take an active role in "fleshing out" the presentation.

One reader who recently looked at this manuscript commented that I should not have said there is no zero in the case of the first-order low-pass (ASP 1-10). Rather, says the reader, there is a zero at infinity. This is a valid argument that can be made, which takes some explaining. We will try to fit this into one of the later issues.

[1] Sallen, R.P., and E.L. Key, "A practical method of designing R.C. active filters," I.R.E. Transactions on Circuit Theory, March 1955.

Electronotes, Vol. 19, No. 191, December 1999

Published by B. Hutchins, 1016 Hanshaw Rd., Ithaca, NY 14850
(607)-257-8010