THE DRUMHEAD PROBLEM - THE VIBRATING MEMBRANE:

-by Bernie Hutchins

Among the percussive sounds that are difficult to synthesize, we find the sounds of various types of drums. Certain well pitched types, such as the bongo, are synthesized without too much difficulty by using ringing filters.

As with any percussive type sound, we do well to first look at how the actual acoustical instrument places its partials. This process usually begins with an idealized form of the instrument, and then we can go on to see how these general mathematical results actually turn out in a practical instrument, if these data are available.

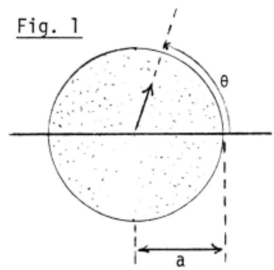
A drumhead is basically a vibrating membrane, and the case of the vibrating membrane is a well studied problem in mathematical physics, and one which is often a textbook example. Unfortunately, there are several problems with these presentations. First, they are often incomplete, being intended to teach mathematics and not the physics of the membrane. Secondly, the most mathematically interesting solution (the one with radial symmetry, leading to Bessel function solutions), is not the one that we find in actual musical instruments. Finally, such a problem has boundary values, and some points about this, while mathematically evident, are not explained physically. Since I was not able to find a satisfactory presentation in the literature, I thought it might be useful to work it all out here, particularly since this would afford the opportunity to pay more attention to the physics involved.

Fig. 1 shows a circular membrane stretched across a circular frame. We assume the membrane to be uniform and to be everywhere under the same tension. Any membrane of this type must obey the differential equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \qquad (1)$$

where u(t) is the up/down motion of any part of the membrane, c^2 is the tension divided by the density in suitable units, and ∇^2 is the Laplacian operator. The derivation is available in many references [1, 2, 3].

Fig. 1



For the circular membrane, we have an obvious case of radial symmetry, so the problem is solved in polar coordinates. Thus we write out the Laplacian operator in its polar form, and equation (1) becomes:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right] \qquad (2)$$

This is a classis "separation of variables" problem. To solve the problem, we will first assume that $u(r,\theta,t)$ is the product of three functions, $R(r)$, $\Theta(\theta)$, and $T(t)$. We need to consider if this assumption is reasonable. We probably have no problem with the time dependence. If the membrane is set into motion in a certain way at a certain time, we expect a resulting response. If we do it in exactly the same way later, we expect the same response. Thus, we are not surprised that the behavior of the membrane at any position has an independent time dependence. The independence of the position coordinates (the distance out from the center, and the angle around the circle) can be understood in terms of the uniformity of the membrane and of the circular symmetry. That is, we have chosen coordinates so that independent functions can be used. [If we had chosen x and y coordinates for example, a position on a circular membrane at a given value of x is not everywhere similar, and we would expect the response to also depend on y. However, using r for example, all points at a given radius are similar.] We need to be sure to note that we must eventually get around to involving boundary (or initial) conditions, once the general from of the solution is determined. These will indeed make certain areas of the membrane special. In fact, it should be clear that the only boundary condition we have at the moment is the restriction that $u(r=a, \theta, t)$ be equal to zero (the edge is fixed). Thus, unless we are going to specify that the membrane is to be excited with non-radial symmetry, we have no need for $\Theta(\theta)$. Of course, we certainly are interested in non-radial excitation. We expect to strike drumheads not just at the middle, or worse still, with some sort of ring structure, but at some point off center. Thus we need to keep the solution general.

We have assumed now that $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$. This greatly simplifies the mathematics since the partial derivatives of equation (2) now apply only to one of the three multiplying functions, and not to the complete function u. If we plug this into equation (2), we get:

$$\Theta T \frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\Theta T \frac{\partial R}{\partial r} + \frac{1}{r^2} RT \frac{\partial^2 \Theta}{\partial \theta^2} = \frac{1}{c^2}\frac{\partial^2 T}{\partial t^2} R\Theta \qquad (3)$$

The usual procedure is then to divide the equation by $R \cdot \Theta \cdot T$, which gives us:

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{rR}\frac{\partial R}{\partial r} + \frac{1}{r^2 \Theta}\frac{\partial^2 \Theta}{\partial \theta^2} = \frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} \qquad (4)$$

We have now separated the variables, and will follow the usual procedure. Observe that we have one side of the equation that depends only on r and θ, and the other side which depends only on t. These can be equal only if each side is equal to the same constant value. This can be seen if you consider that neither side has a means of "controlling" the other, so the equality can hold only if some other factor is restraining the situation. We can call this constant anything we choose, but it is the usual practice to choose a symbol that will prove useful later. Thus we will choose the constant to be $-(\omega_{mn}/c)^2$ where I hope no reader will be surprised to see that ω_{mn} will turn out to be a frequency of interest to us.

First we set the right side of our equation equal to the constant:

$$\frac{1}{T}\frac{1}{c^2}\frac{\partial^2 T}{\partial t^2} = -\frac{\omega_{mn}^2}{c^2} \qquad (5)$$

This is just the equation of simple harmonic motion, and can be written in the form:

$$\frac{\partial^2 T}{\partial t^2} + \omega_{mn}^2 T = 0 \qquad (6)$$

which has a solution

$$T(t) = \mathrm{Cos}(\omega_{mn} t + \phi_{mn}) \qquad (7)$$

Here we are neglecting the amplitude term which should multiply any general solution, but this will appear automatically in the total solution as part of R(r). The phase term ϕ_{mn} is an arbitrary starting phase that will depend on the exact starting conditions, and is not all that important.

What do we have so far. We have equation (7) which tells us that the time behavior of $u(r, \theta, t)$ is just simple harmonic motion at a frequency ω_{mn}. Any portion of the membrane is just moving up and down in a sinusoidal manner. Yet we are far from finished. For one thing, the amplitude of this motion will be determined by the two spatial coordinates r and θ, so we need to solve the remaining two parts of the equation. This amplitude will be different for different parts of the membrane. We also have not determined an expression for ω_{mn}. And as one might gather from our use of the subscripts m and n on ω, we are expecting more than one frequency to be a solution, and any general solution will be the superposition of a number of the possible vibrational modes available. In fact, the m part of the solution will come from the angle, while the n part will come from the radius.

We now have to remember where we were. We have set the right side of equation (4) equal to our constant, and it is now time to work on the left side. Let's do a couple of things at once. We will set this left side equal to the constant, and then move the constant to the left side, while at the same time moving the term that depends on θ to the right side. This gives:

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{rR}\frac{\partial R}{\partial r} + \frac{\omega_{mn}^2}{c^2} = -\frac{1}{r^2}\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} \tag{8}$$

This we can multiply through by r^2, which will remove the r dependence from the right side of the equation. Thus, as with equation (4), we will have separated the variables and it must be true that both sides are equal to a constant. We will call this constant m^2, and it will turn out that we can relate this m to the m subscript on ω_{mn}, and this will be established later. However, note that at the moment we have not restricted m in any way. Completing the operations suggested above, we have:

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\frac{\partial R}{\partial r} + \frac{\omega_{mn}^2 r^2}{c^2} = -\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = m^2 \tag{9}$$

Leaving the left side of equation (9) for later, we can work with the remaining parts, giving us:

$$\frac{\partial^2\Theta}{\partial\theta^2} + m^2\Theta = 0 \tag{10}$$

which is identical in form to equation (6), and we have again the simple harmonic motion solution:

$$\Theta(\theta) = \mathrm{Cos}(m\theta + \theta_o) \tag{11}$$

Actually, m has not been specified exactly, but we are here forced to the conclusion that m must be an integer. This is because θ, being an angle around the circle, will take us back to the same angle for any multiple of 2π that may be added to it. Thus unless m is an integer, we will not arrive at $\Theta(\theta) = \Theta(\theta + 2\pi)$. We might imagine that it might be possible for m to be a non-integer, but basically, this is not in the mathematics, as we have assumed all the time dependence is in T(t). Thus we take m to be an integer, and θ_o becomes our reference angle for θ.

All that really remains of our solution now is the left side of equation (9) and this is the most difficult part mathematically. Setting this left side equal to the constant m^2 and moving this constant to the left side, we arrive at:

$$r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + \left[\frac{\omega_{mn}^2 r^2}{c^2} - m^2\right]R = 0 \tag{12}$$

This is obviously a rather imposing equation, at least as compared to equations (6) and (10). Fortunately, the equation can be recognized as Bessel's differential equation, and the solutions are the well-known Bessel functions. We have only to look up the solution in a handbook [4] and write it down. This gives us:

$$R(r) = C_{mn}J_m\ (\omega_{mn}r/c) \tag{13}$$

Here C_{mn} is an overall amplitude function for the mode of vibration in question, J_m is the m^{th} order Bessel function of the first kind, and the remaining terms are as used above. We can now write the complete solution $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$ using equations (7), (11), and (13) as:

$$u(r,\theta,t) = C_{mn}J_m(\omega_{mn}r/c)\mathrm{Cos}(m\theta + \theta_o)\mathrm{Cos}(\omega_{mn}t + \phi_{mn}) \tag{14}$$

Now, having the complete solution we can apply any boundary conditions we have to find the actual vibrational frequencies. Note in particular that the ω_{mn} from the time dependent equation and the m from the angle dependent equation have found their way into the radial solution. This is a fortunate situation, as our only real boundary condition involves the radial part: we know that at the circumference, at $r = a$, the amplitude of the vibration must go to zero, as it is clamped there. Thus we can write:

$$u(a, \theta, t) = 0 \quad (15)$$

This is only possible if the part of the solution which involves r, $R(r)$, is equal to zero, since neither $\Theta(\theta)$ nor $T(t)$ can be everywhere zero for a special value of r. Examination of equation (14) thus shows us that we set $r = a$, and then it must be true that:

$$J(\omega_{mn}a/c) = 0 \quad (16)$$

which means in turn that:

$$\omega_{mn} = (c/a)\ z_{mn} \quad (17)$$

where z_{mn} is the n^{th} zero of J_m. Thus we finally get our most important result. Equation (17) tells us that the vibrational frequencies are known if we just look up the zeros in the Bessel functions J_m. A table of these is shown below [5]:

Zeros of J_m TABLE 1

	J_0	J_1	J_2	J_3	J_4	J_5
n=1	2.404	3.832	5.135	6.379	7.586	8.780
n=2	5.520	7.016	8.417	9.760	11.064	12.339
n=3	8.654	10.173	11.620	13.017	14.373	15.700
n=4	11.792	13.323	14.796	16.224	17.616	18.982
n=5	14.931	16.470	17.960	19.410	20.827	22.220
n=6	18.071	19.616	21.117	22.583	24.018	25.431
n=7	21.212	22.760	24.270	25.749	27.200	28.628
n=8	24.353	25.903	27.421	28.909	30.371	31.813
n=9	27.494	29.047	30.571	32.050	33.512	34.983

It is also useful to have a table of these zeros all divided by the first zero of J_0. Below we show the equivalent of Table 1 with all the values divided by 2.404.

Zeros of J_m divided by 2.404 TABLE 2

	J_0	J_1	J_2	J_3	J_4	J_5
n=1	1.000	1.594	2.136	2.653	3.156	3.652
n=2	2.296	2.918	3.501	4.060	4.602	5.133
n=3	3.600	4.232	4.834	5.415	5.979	6.531
n=4	4.905	5.542	6.155	6.749	7.328	7.896
n=5	6.211	6.851	7.471	8.074	8.663	9.243
n=6	7.517	8.160	8.784	9.394	9.991	10.579
n=7	8.824	9.468	10.096	10.711	11.314	11.908
n=8	10.130	10.775	11.406	12.025	12.634	13.233
n=9	11.437	12.083	12.717	13.332	13.940	14.552

A graph of some of the Bessel functions is shown in Fig. 2.

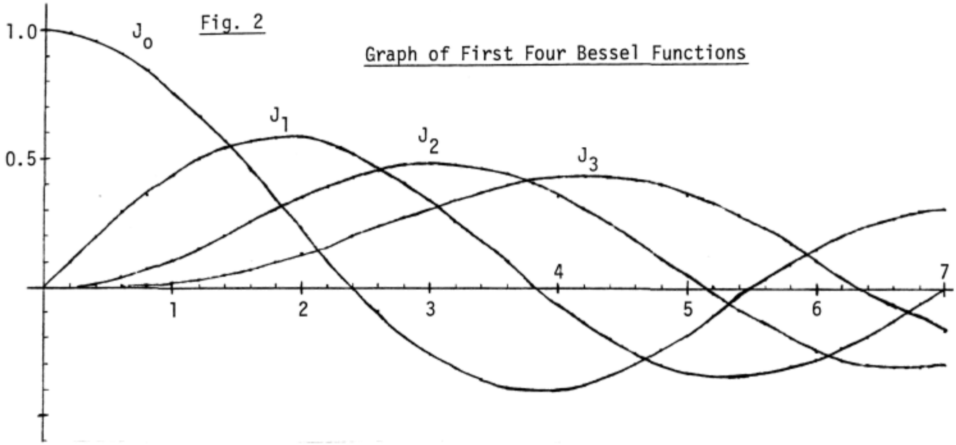


Fig. 2

Graph of First Four Bessel Functions

With any luck, we have not lost sight of the physical results here, and we are about to look at these physical results in more detail. Note that Table 2 gives the frequencies of all possible modes (within the limits of the table) of the membrane, relative to the lowest frequency (J_0 for n=1). The physical significance of the Bessel function shapes of Fig. 2 is that they represent a cross section of the membrane for the various cases where they appear. More on all this later.

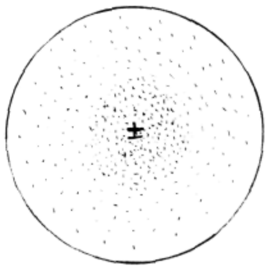
We need to represent the modes of the membrane somehow, and this will be done with circles, as shown in Fig. 3. Any solid line on these diargams, whether it is straight or a circle, represents a line where the motion of the membrane is zero. Thus we have an outside circle where we have assumed that the membrane is clamped. There are also lines on the membrane inside the boundary where the motion is also zero for a given mode. These are called "nodes." Nodes can be produced only by the $R(r)$ and $\Theta(\theta)$ functions. It is true that the $T(t)$ function can cause the membrane to reach zero displacement, but this is not true for all time, but only instantaneously. Nodes occur only when the "amplitude" function $R(r)\Theta(\theta)$ for $T(t)$ is a zero. The pattern of nodes for a given mode depends on its order.

Consider first equation (11). As θ goes from 0 to 2π, for m=1, we find two values, namely 0 ($=2\pi$) and π where $\Theta(\theta)$ becomes zero. This means that there is a node for these angles. Clearly this represents a node on all of a diameter of the circle. If m=2, there are two nodal diameters, and for a general value of m, we have m nodal diameters.

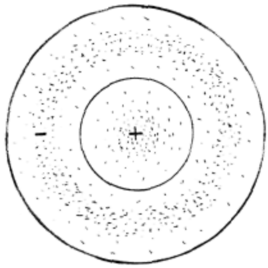
We also get nodal circles as well as nodal diameters, and these come from zeros in the $R(r)$ function. Our boundary condition tells us that one of the zeros of J_m must be at r=a. Consider first the case where m=0, in which case we have the membrane in the shape of J_0. The boundary could be fixed at any of the zeros, at 2.404, at 5.520, etc. Perhaps it is a better point of view to say that the scale of the Bessel function argument must expand or contract to fit so that a zero occurs at r = a. In either case, nodal circles do occur and appear at zeros of J_m.

In making our drawings, it is necessary to show in addition to the nodal lines, the relative motion of various sections between the nodes. This will be done by showing out of phase sections by (+) signs and by (-) signs. It should be understood however that these sections are not permanently "up" or permanently "down", as they are oscillating up and down according to $T(t)$. Rather it just indicates a difference of phase.

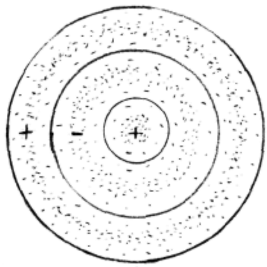
Finally, we would like to be able to show a difference of amplitude for different parts of the membrane, and this we will indicate by a greater density of dotting for greater amplitudes. Thus we can see in diagrams such as those of Fig. 3 that there is a low density of dots in the vicinity of any node, and a maximum as we move as far as possible from nodes. Note also that when we cross a node from one area to another, we end up in a region of the opposite sign. This is simply because the membrane must cross a node straight across, and not come down to zero and return in the same direction. This latter case would require a nodal region with the membrane acting at all times to pull the membrane away from the node. We are now in a position to understand the various modes of Fig. 3, and to then go on to some additional points of interest.



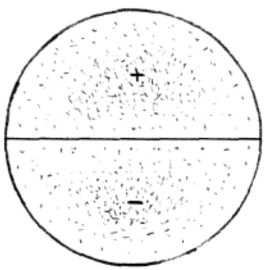
Mode 0,1
Freq. = 1.000



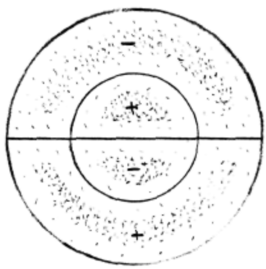
Mode 0,2
Freq. = 2.296



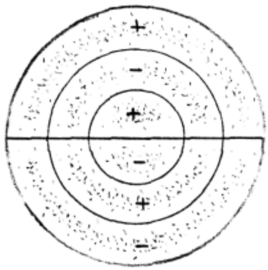
Mode 0,3
Freq. = 3.600



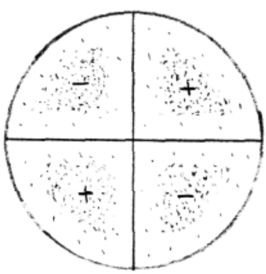
Mode 1,1
Freq. = 1.594



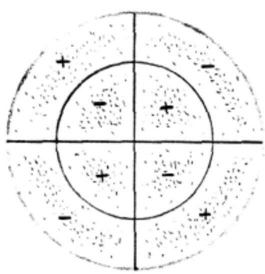
Mode 1,2
Freq. = 2.918



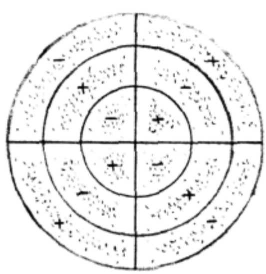
Mode 1,3
Freq. = 4.232



Mode 2,1
Freq. = 2.136



Mode 2,2
Freq. = 3.501



Mode 2,3
Freq. = 4.834

We can now examine the various modes from Fig. 3. Here we show only nine of the theoretically infinite number of modes possible. The modes are denoted m,n where m is the order of the Bessel function J_m, and is equal to the number of nodal diameters, as we have noted. For a given number of nodal diameters, n then denotes the number of circular nodes, counting the outside boundary at r = a, which is a perfectly good node. The three top modes in Fig. 3 represent the case of no nodal diameters. The mode 0,1 is the simplest motion. Here the membrane just moves up and down, more at the center than the edges, but as a unit. The limits of its motion, or its shape at any instant in cross-section, are given by the curve of J_0 on the interval from 0 to 2.4 (see Fig. 2). The mode 0,2 has a circular node. Now the outside clamping is found in the second zero of J_0 which is at 5.52. Thus there is an inside node at 2.404/5.52 = 0.44a. Note that as far as the inside part of the membrane is concerned, it is responding in a 0,1 mode with a radius of 0.44a. It does not know nor care what is going on beyond 0.44a since there is a node there. The motion of the 0,2 mode is suggested more completely in cross section as shown in Fig. 4. Note that we must consider the boundary at r=a to be "hinged" in order that it act just as a node does at r=0.44a in the 0,2 mode. This assumption of a hinged edge is implied by an assumption of a totally flexible (no stiffness) membrane. Fig. 5 shows a comparison of a hinged edge and a transition across a true node. The mode 0,3 is similar. This mode represents J_0 from 0 to 8.654, with additional nodal circles circles at 5.52 and 2.4. Scaled to the 0 to r=a radius of the membrane, these nodal circles occur at 5.52/8.654 = 0.64a and at 2.4/8.654 = 0.28a.

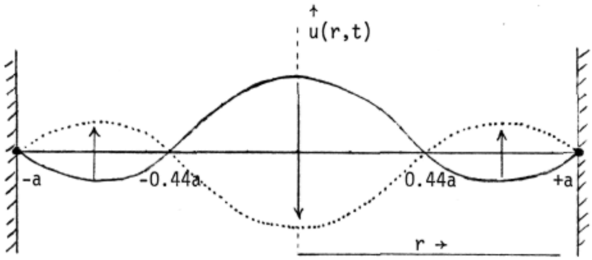


Fig. 4

Motion for mode 0,2

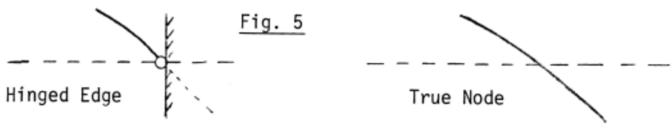


Fig. 5

Next we look at modes with m greater than zero. These modes all have one or more nodal diameters. Note that since the diameter naturally passes through the center of the membrane, and is everywhere zero on that diameter, there must be a zero in the center. Thus we can understand how it is that J_0 can no longer be involved in the solution (see Fig. 2). In a bit, we shall discuss why it is that a particular Bessel function is physically chosen, since it is obvious that any J_m for m greater than zero will meet the center zero condition imposed by any number of nodal diameters.

Mode 1,1 has one nodal diameter, and the outside nodal circle. According to the complete solution, equation (14), we see that J_1 is the Bessel function chosen here. J_1 has a zero at 3.832, and thus the portion of J_1 in Fig. 2 between 0 and 3.832 can be considered a cross-section of the membrane - anywhere except on the nodal diameter. If this cross-section is taken at a small angle to the nodal diameter, we expect its amplitude to be fairly small, with the maximum amplitude occuring when the cross-section is taken perpendicular to the nodal diameter. Note that the nodal diameter breaks up the radial symmetry of the m=0 modes that was seen above.

The setup of the modes 1,2 and 1,3 in Fig. 3 is similar to that for the m=0 modes. Note however that we are dealing with J_1 here. Thus the 1,2 mode has its nodal circle at 1.594/2.918 = 0.55a, which is a larger radius than for the 0,2 mode. The radii for the 1,3 mode are obtained in the same manner, dividing 1.594 and 2.918 by 4.232 in this case. [All data from Table 2]

Nothing really new is found when we go to the m=2 modes of Fig. 3, except here we have two nodal diameters. The mathematics tells us that we must go to the J_2 function now. The physical reason for this is not clear immediately however, as J_1 (or J_3, or any higher J_m) would also meet the restriction of having a zero at r=0. So why is it J_2 that is chosen? To get some idea about this, Fig. 6 is provided. Here we have taken the portions of J_1, J_2, and J_3 that lies between 0 and their first zeros, and have compressed all to a length of 1, and also normalized the amplitude to unity. Note that all three have the necessary zeros at x=0 and at x=1. So why prefer J_2? Consider that for mode 2,1, that the x=0 portion of the curve occurs at the center, which is a 90° angle for that nodal segment, while the x=1 part of the curve occurs at the outside of the circle, which is, relatively speaking, almost a straight line. We expect the membrane in the pointy region at the center to resist bending upward more than the corresponding region near the edge. Thus a curve such as J_1 in Fig. 6, which bends upward to much the same degree at either end, is not as suitable as would be J_2, which bends up gradually at the x=0 end. Evidently, J_3 must be too gradual a bend up at the x=0 end for this case, and correspondingly comes down too sharply at x=1. Yet, as seems reasonable, in the case of three nodal diameters (not shown in Fig. 3), we would have a 60° angle at the center, even more pointy, and here, the function J_3 is the solution selected by nature. Thus while we do not offer here an exact proof from a physical basis that J_m is the proper choice for m nodal diameters, we have shown that it is physically reasonable. And of course, the mathematics demonstrates the correct result.

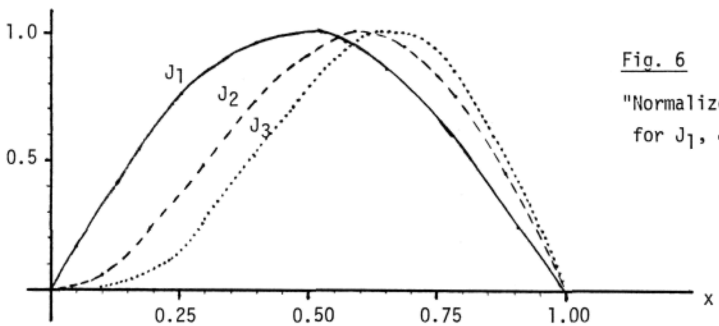


Fig. 6

"Normalized" curves for J_1, J_2, and J_3

This pretty well completes our study of the ideal vibrating membrane. We need to make a few additional comments however, the most important of which is with regard to superposition. Any complete solution of the vibration problem would include a set of initial displacement and velocities for various parts of the membrane. The solution in such a case would involve a superposition of modes. That is, we would not have just one of the possible modes, as in Fig. 3 for example, but a whole number all on top of one another. We must be clear what this means. Each of the individual modes can be considered independent. Thus while a given mode may have a nodal diameter or a nodal circle at a given location, this does not mean that the membrane is not actually moving at that point. A different mode may be driving it at that point. It is in fact the case that some care must be taken to excite only a single mode, the general case being the excitation of several or many modes. To excite only one mode requires the set-up of exact initial conditions or the placement of dampers.

Dispite the very complex total vibration of superimposed modes, it is still true that each of the individual modes can be considered independent, and that each mode actually excited could be heard as sound coming from the membrane. Since musically it is this sound that we are interested in, not the interesting vibrational patterns, our main interest is in the ratios of the frequencies produced. Clearly they are extremely non-harmonic in nature. As a final note before going on to consider how all this actually applies to a real drum, consider that we have solved a wide variety of problems in addition to the circular membrane, although these may not be useful. We have seen that any region surrounded by nodes could be taken to be clamped at the boundary. Thus, for example, we know how to handle the vibration of a quadrant of a circle, as part of the 2,1 mode of the circle.

A timpani or any other drum is not an ideal vibrating membrane. In fact, even a real membrane clamped at the perimeter is not an ideal membrane. As we have seen, the natural modes of vibration are not hramonically related for the ideal membrane. Compare this with the ideal vibrating string where the modes are harmonics. It is often the case with percussion instruments that the "prototype" mechanican structure is not harmonic, but that craftsmen through the ages have made empirical modifications that make some of the more important modes harmonics or near-harmonics. In fact, a study of musical instruments leads us to wonder just how these craftsmen could have accomplished these adjustments which even today do not yield well to mathematical analysis.

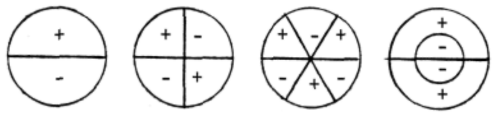
We want to look briefly at the kettledrum. [A timpani is more than one kettledrum by the way.] On the top is a vibrating membrane of calfskin or plastic. It is secured at the perimeter. Yet, it has below it the traditional copper-colored kettle or bowl shaped container. Why is this here? Well, of all the possible suggestions for the kettle, it is most probably used because it alters the vibrational modes of the membrane on top. We would expect this, as the kettle encloses a volume of air, and this air is springy, adding to the restoring forces in the membrane itself. Actually, there is an air hole or vent in the kettle. This we would need, if for no other reason, to keep the kettledrum from acting as a thermometer/barometer. Yet the hole is about an inch in diameter, somewhat larger than would be necessary if it were just for pressure equalization. In fact, it does seem to have an optimum size for the further tuning of vibrational modes.

Naturally, the best way to approach this problem would be to first take some data. Data on the kettledrum have been taken by Rayleigh [5] and by Benade [6], and the results tell us which modes are excited, and how their frequencies compare with the theoretical ones of the ideal membrane. First, it turns out that the modes that are excited the most are the 1,1, the 2,1, the 3,1, and the 1,2. These are illustrated in Fig. 7 below. Note in particular that all these have nodal diameters. The modes with radial symmetry are not excited well, probably because they would have to move too much air behind them, and the kettle and the dimensions of the membrane restrict this. Because these modes are not excited well, we find that the playing technique is to strike the drum membrane nearer the edge than toward the center, and thus we can understand the excitation of modes with nodal diameters.

Fig. 7

Mode	1,1	2,1	3,1	1,2

Pattern



Ideal Frequency Relative to 0,1	1.59	2.14	2.65	2.92
Ideal Frequency Relative to 1,1	1.00	1.34	1.66	1.83
Measured Frequency (Rayleigh [5])	1.00	1.50	1.89	
Measured Frequency (Benade [6])	1.00	1.504	1.742	2.000

The data are clear from Fig. 7. The major thing that has happened is that the vibrational frequencies have been moved from their ideal membrane positions to positions that are much more nearly harmonic. The best data is probably Benade's, and he shows nearly exact harmonics of the lowest excited mode at the fifth (1.5) and at the octave (2.0), with the 3,1 mode being a sharp sixth. So instrument design, tuning, and skillful playing result in an instrument with close harmonics. Yet not all the partials are, or need be harmonics. We expect that the character of percussion instruments has something to do with non-harmonic partical.

All the above is interesting an a mathematical exercise for understanding how physics governs vibration, and as an aid in helping us to appreciate the musical instrument craft. The results are somewhat typical of other traditional musical instruments in their complexity and in presention of imperical modifications toward a more harmonic instrument.

One final point of interest can be made. What does an ideal membrane sound like? For that matter, what does an ideal string sound like? Well, we know what an ideal string sounds like - it is purely harmonic, and any periodic waveform we might consider is one possible example of an ideal string. Thus to hear a perfect string, we listen to an electronic simulation. We certainly have no perfect mechanical membranes. However, we could make a perfect electronic membrane. We would simply arrange a set of sine wave oscillators at the frequencies of the ideal membrane, and assign to them arbitrary amplitudes, and then just listen and experiment. This is probably a good project for computer simulation.

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LeGendre Filters:

-by Normand Provencher

As stated in AN-207 (2), the amplitude response of an n^{th} order Butterworth low-pass filter is of the form:

$$A(\omega) = \frac{A_0}{[1+\omega^{2n}]^{1/2}}$$

where n, a positive integer, is the order of the filter. The expression $1 + \omega^{2n}$ is called the Butterworth approximation function. Approximation functions are various functions in ω chosen to match as closely as possible an idealized amplitude response which in the case of low-pass filters, consists of a flat amplitude response up to the cut-off frequency, followed by infinite attenuation of all frequencies larger than the cut-off frequency. In general, the amplitude response of a low-pass filter can be written as

$$A(\omega) = \frac{A_0}{[1+f(\omega^2)]^{1/2}}$$

where $f(\omega^2)$ is a positive rational function of ω^2; if the function has no finite zeros then $f(\omega^2)$ is a polynomial. Two well known types of polynomial filters are the Butterworth given by $f(\omega) = \omega^{2n}$ and the Chebyshev given by

$$f(\omega^2) = \tfrac{1}{2}C_n(2\omega^2 - 1) + \tfrac{1}{2} = C_n^2(\omega)$$