ELECTRONOTES
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APPLICATION NOTE NO. 430

Sept 27, 2016

FEEDBACK AND SENSITIVITY

INTRODUCTION:

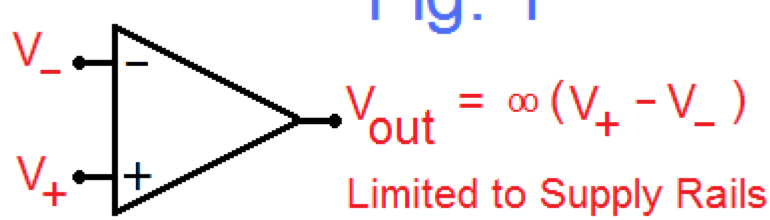
Here we will think of a feedback system as a circuit having a signal from some output point back to some input point. The subject is important and fairly well understood [1]. The feedback might be accidental (such as a positive feedback from a PA speaker feeding back to a microphone) or intentionally built into an audio amplifier (such as negative feedback to flatten a gain curve and/or reduce distortion). It may even be part of something like a climate model. When the path parameters are known so that full multiplier factors and phases are known, it is generally possible to determine if a gain of some device increases or decreases as a result of the feedback.

We want to be sure we understand this calculation. Beyond this, we want to consider how to calculate the effects of an inexact knowledge of the feedback parameters in an actual case – the so-called sensitivity problem. The general sensitivity problem is also well understood [2a, 2b, 2c].

IT BEGINS WITH AN IDEAL OP-AMP

Our study begins with the op-amp which is short for operational amplifier. Engineers deal with op-amps that are real, but also often start with op-amps that are considered ideal, always mindful of the fact that reality may well come in to change our success. On the other hand, many practical circuits are based on ideal assumptions and are near-nominal. Here we shall be using ideal op-amps for theory and for examples, and to model feedback situations. Experiments to verify the results are essential, and of course done with real op-amps.

Fig. 1



Ideal Op-Amp

Fig. 1 shows an ideal op-amp and gives its basic equation. The op-amp is ideally a 3-terminal device [in practice, it also has two power supply connections, almost always not shown for an ideal case, and often omitted (understood to exist) in a practical diagram]. Note that there are two inputs, one V_- being an inverting input and the other V_+ being a non-inverting input (essentially a double negative). The output voltage is given by:

$$V_{out} = \infty\ (V_+ - V_-) \qquad (1)$$

This is indeed a strange equation since it has a multiply by ∞. Notice that Fig. 1 gives the additional stipulation (reality) that the output is limited to the power supply limits, usually something like ±15 volts. Equation (1) is not too useful as such – but it tells us that either the output is at +15 or at -15, OR ELSE the differential input $(V_+ - V_-)$ must be zero (the strange case of 0•∞ being a finite value between +15 and -15). This "miracle" of achieving zero differential input is achieved with negative feedback by the op-amp itself. That is, this note will be mainly (ultimately) concerned with positive and negative feedback added to usual op-amp configurations (amplifiers, summers) which are realized with op-amps by well known and time-honored negative feedback methods.

About the only other properties of the ideal op-amp to note are that the inputs are assumed to draw no currents (infinite input impedance) and the output can supply any current without changing its voltage (zero output impedance). Appendix A lists corresponding properties of ideal and real op-amps. As suggested, many (not all) real circuits are ideal for practical purposes. The experimental op-amp circuits here confirm the ideal op-amp theory.

OP-AMP CIRCUITS

At this point we need to take care to avoid a diversion into a discussion of the cornucopia of op-amp circuits that are common in the design art. In consequence, here we assume this is familiar to the reader or is left as an exercise. The essential point at this stage is that feedback is already in use in the configurations we are going to study and build with. In the one sense we may be essentially involved with changing existing feedback in the building blocks. On the other hand, we may be illustrating a feedback flow graph. The emphasis is in making a point. This often means that the circuits we show will not be efficient designs. A more direct approach might well involve fewer op-amps and fewer resistors to the same function (perhaps a gain of +3). That is, the somewhat cumbersome networks may be illustrating the points with clearer functional blocks (single function) rather than with a circuit we would actually build or even one we might consider "clever" to impress the reader. Basic brute-force stuff here.

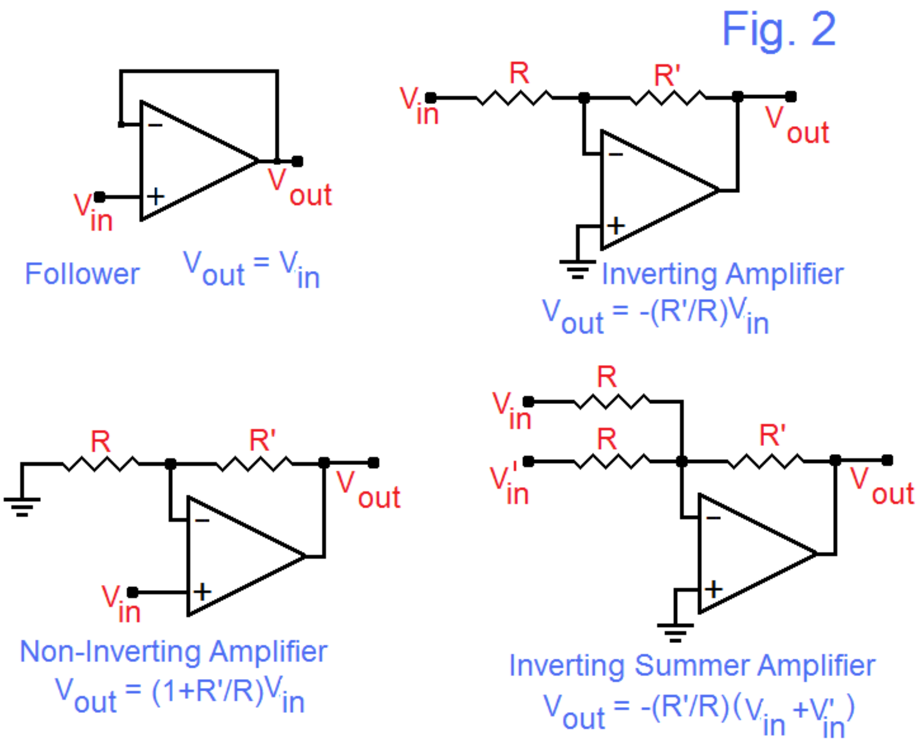


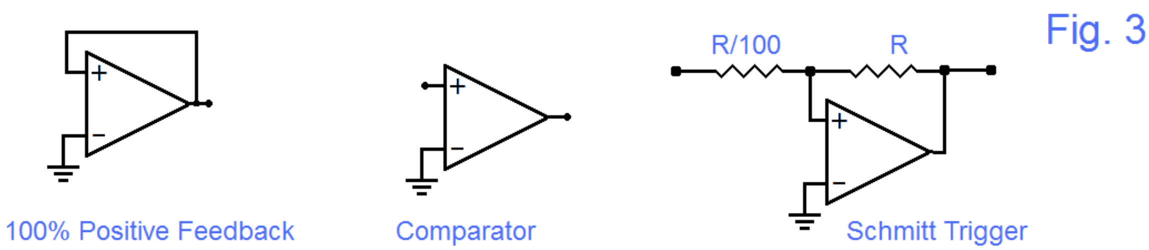
Fig. 2 shows what are likely the four most fundamental negative feedback op-amp circuits: the follower, the inverter, the non-inverter, and the (inverting) summer. Most likely these are familiar, or are easily derived using equation (1); that is $(V_+ = V_-)$, and then Ohm's Law and current summing, etc. Some hints are offered in Appendix B, and full discussions are easy to locate.

An interesting notion is whether or not any one of these op-amps "knows" it is in a feedback configuration. Like, doesn't the op-amp in the follower recognize (meaning, I guess, behave in a particular way) because its own output is also its (-) input? Like – doesn't it "catch on" to what is going on! Absolutely not. It just obeys equation (1) at all times.

These circuits are important because we have amplifiers, inverters, scalars, and summers here – all we need for feedback loop studies. These "building blocks" all use negative feedback just to function, but the subject of feedback, as is under discussion here, has not been intruduced yet.

A BIT ON OP-AMPS WITH POSITIVE FEEDBACK

Bare op-amps with positive feedback also are common, but for application as comparators or Schmitt triggers – things we need, but distinctly different from the negative feedback devices of Fig. 2. And the positive feedback ideas for these devices



are distinctly different from the way positive feedback works to boost gain in the main theme of this note.

Above we said that the differential input needs to be zero or else the infinite gain drives the output to +15 or -15. As noted, the op-amp "has no idea" where its input voltages are coming from. But it is only through negative feedback properly working that $(V_+ = V_-)$. This can fail to occur. (For example, if the output is amplified to exceed the supply limits, negative feedback can fail. Another case is when there is a reactive elements (meaning, usually, capacitors) such that a phase shift makes negative feedback positive.) Here we explicitly use the (+) input for a positive feedback. Three cases are shown in Fig. 3.

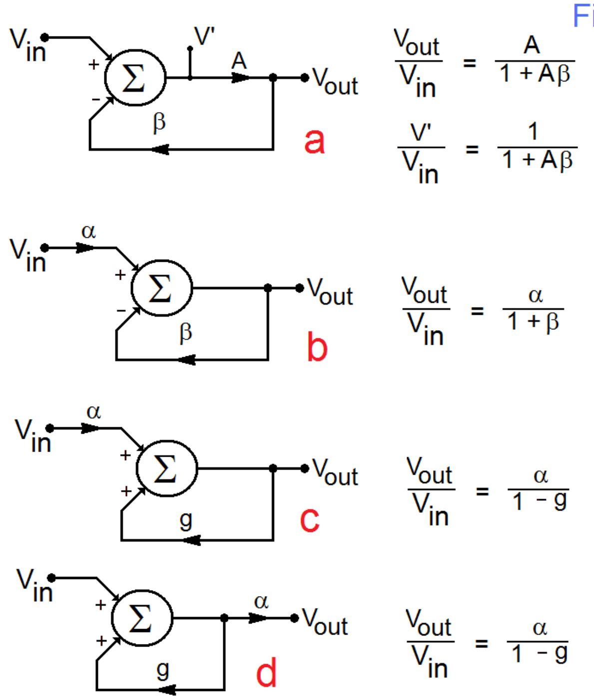
In Fig. 3 (left) we have a curiosity where the output is fed to the (+) input with the (-) input grounded. Clearly the op-amp is happy if the output is +15 or -15, causing the differential input (non-zero) to have the correct sign. Which is it? It depends on how it comes up when power is applied. Note that if you take a wire and briefly touch the output to +15 or to -15, and that forces the choice. Fig. 3 (middle) is a classic comparator. There is no feedback here but the circuit is useful and leads to the Schmitt trigger. The output is +15 if the input is above zero and -15 if the input is below zero. What if it is zero? Well, we don't usually expect to achieve this, but we can suppose that it might cause the output to "chatter" back and forth. The Schmitt trigger adds some "memory" or "hysteresis" to the comparator, so that once it flips self-reinforces the flip until some large enough and defined change of input occurs.

So we have practical and simple op-amp circuit using both negative and positive feedbacks. The main theme here is the case where we want illustrative blocks with known finite gains between loops.

FEEDBACK ALTERING THE GAIN OF A FINITE GAIN AMPLIFIER

When it comes to feedback around some loop, positive or negative, we need to pay close attention to signs and where a signal is being put in and where it is taken out. A

Fig. 4



major stumbling point is trying to look at some one textbook feedback diagram as being "universal" (along with a corresponding equation) rather than just simply working out the existing circuit and equation. That is, derive anew (trivially) instead of adapting existing examples.

Fig. 4 shows just four of perhaps a dozen variations on a feedback network, along with input/output equations. The flow-graph in (4a) is probably the most classis, a negative feedback of β around an amplifier of gain A. We simply write down the equation for V_{out}.

$$V_{out} = A(V_{in} - \beta V_{out}) \tag{2}$$

which is solved for

$$G = \frac{V_{out}}{V_{in}} = \frac{A}{1+A\beta} \tag{3}$$

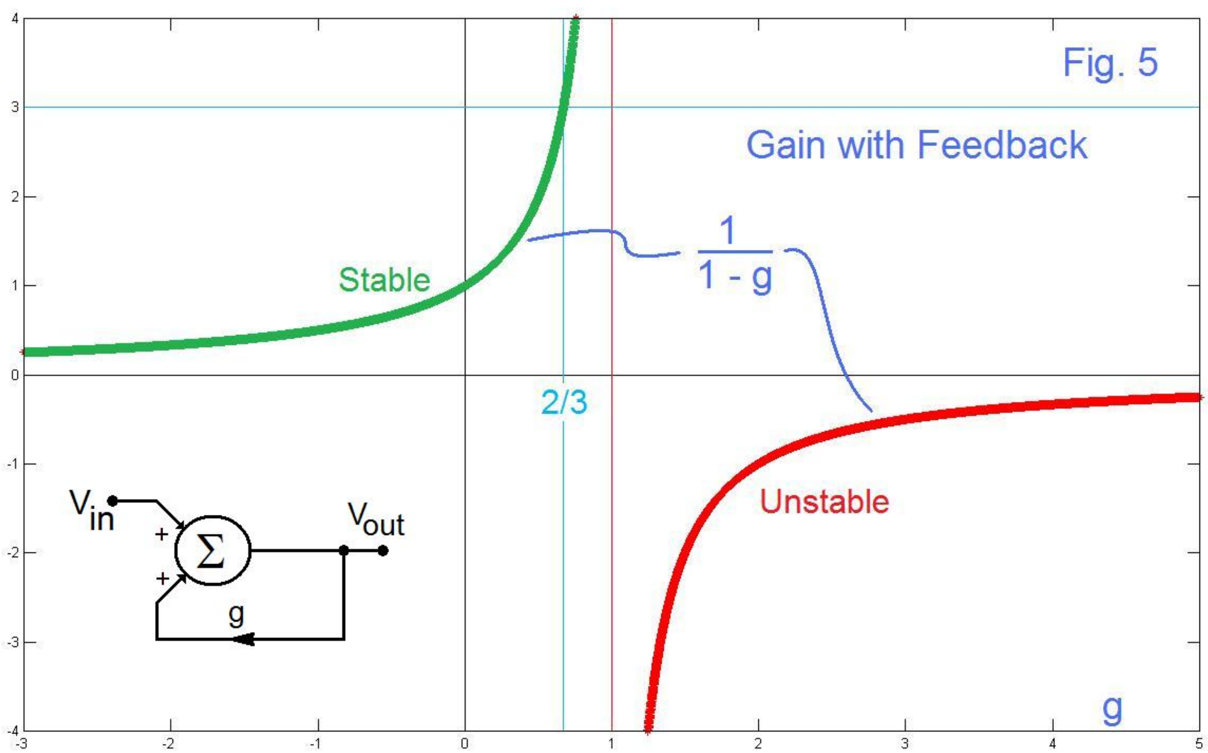
This is trivial of course. It could get complicated because A and/or β might be not just constants, but function of frequency. In this note, we will take both to be constants.

For example, suppose A is 100 and β=0.05. In this case, we have negative feedback around an amplifier A and G becomes 16.67. That is, the gain is reduced (from 100) to a somewhat smaller value. G is called, appropriately, the "closed-loop gain". Note that if β = 0 the feedback loop is broken, and G becomes A, the original gain, called the "open-loop gain". To complete the terminology, Aβ, is called the "loop gain" (the total gain around the loop). (See Appendix C).

So we just fit this model? Then what happens if a (+) rather than a (-) input is used for the loop? Equivalently, what if β becomes negative? What if A is just a "piece of wire"; so A=1. What if A is negative? What if A is outside the loop? So many options suggests the wisdom of drawing a correct flow-graph and calculating from it, the correct equation.

Fig. 4a, which assumes A is positive and negative feedback is employed, is thought of as the starting point because traditionally, negative feedback allows for a well-controlled gain, a flatter response, less non-linear distortion, and more favorable input and output impedances. The classical advantages of negative feedback in modern amplifier designs. Kind of like giving up a high gain so that everything else gets better.

The first obvious variation is to consider positive feedback. What if A = 100 and β=-0.05 [still using the (-) input in Fig. 4a]. If we plug into equation (3) we get G=-25. Nothing is <u>obviously</u> wrong. It does look however like an inversion and a severe attenuation, something we are not likely to favor. To see why it's worse than just being <u>not</u> useful, consider that we reduced β from 0.05 through 0 (G=A) and on to β=-0.05 giving a negative G. Note that for a particular value of β=-0.01, G in fact went to infinity as Aβ=1. The function G has a singularity as it goes negative. This is more clear if we let A go to 1 and call Aβ=g (Fig. 4c or 4d with α=1). This gives simply (note the positive polarity of the feedback summer paths here):



$$G = \frac{1}{1-g} \tag{4}$$

This equation is plotted in Fig. 5. Note the singularity at g=1 and the unstable regions when g>1. In agreement with the corresponding calculations using equation (3), values of g up to, but not including 1 (which would be an oscillator) are stable. For example, in Fig. 5, the blue lines show g=2/3 (positive feedback greater than 0 but less than 1) giving a gain G=3. Equation (4) will be easiest to work with going further.

CLASSIC SENSITIVITY AND TUNING EQUATIONS

In many (most?) design problems, various components that determine specific performance are not exact. For example a resistor in an electrical circuit may have specifications that say that its value is a certain nominal value to within a certain manufacturing uncertainty (tolerance) such as 5% or perhaps 1%. When one actually builds the circuit, are the performance parameters off by this same tolerance? Not in general. They might be off by the specified tolerance, or by a larger value, or even by a smaller value, or even not at all. For example the "Q" of a filter might not depend at all on the value of a component that determines cutoff frequency, and vice versa. Getting control over (at least knowledge of) this "Classic Sensitivity" problem is important, particularly in cases of very high sensitivity.

The first useful notion is to make choices (such as among alternative realizations - configurations) and choose one that has generally low sensitivities to passive components (resistors and capacitors). It is not unexpected that the best passive sensitivity results are often associated with high negative feedback gains, with their own "active sensitivity" consequences as a trade-off.

In addition, and very usefully, this mathematical procedure of "Classical Sensitivity" generally permits us to write down "tuning equations". For example, if the filter Q is low by a measured amount, we can calculate quite accurately what change is needed, and we can achieve this correction with impressive precision, for example with small "tweaking" series or shunt resistors (like adding 100 ohms in series to a 10k resistor).

The sensitivity calculations begin with design equations (such as for Q and cutoff frequencies) as algebraic functions of various resistors and capacitors. That is, we have X = X(Y) where Y is perfectly known in theory. We know the formula. The Classical Sensitivity is given as (See Appendix D for filter examples):

$$S_Y^X = \frac{\frac{\Delta X}{X}}{\frac{\Delta Y}{Y}} \approx \frac{Y}{X}\frac{\partial X}{\partial Y} \qquad (5)$$

The middle term shows this as the ratio of the fractional change is X to the fractional change in Y. This is the sort of thing you would ask for, even if you don't study calculus: if Y changes by 5%, how much does X change. If it changes by 5% too, you feel you are breaking even: S_Y^X = 1 . But it can be more or less, perhaps 4, or 1/2, or -1/2. But it does not always come out to be a constant – it may depend on X (thus on Y by the design equation). That is S_Y^X may be a function of X, the particular "operating point" or design value. In that case, you plug the nominal design value into the equation. In a filter, for example, the Q might be insensitive at low values of Q and very sensitive for high values. Where the equation does not give a constant, we need to plug in the neighborhood where we intend to use the device.

As an example, suppose X = 5Y. Simple enough. That is, considering that derivative dX/dY (or the partial derivative $\frac{\partial X}{\partial Y}$) = 5, (the slope), it might appear that the error would be 5 times worse. But clearly the percentage changes will be the same, as we see from:

$$S_Y^X = \frac{Y}{X}\frac{\partial X}{\partial Y} = \frac{Y}{5Y}\,5 = 1 \qquad (6a)$$

However, you can see that if X = Y^2,

$$S_Y^X = 2 \qquad (6b)$$

which is more sensitive, or if X = $\sqrt{Y}$, then

$$S_Y^X = \frac{1}{2} \qquad (6c)$$

which is less sensitive.

A case where the sensitivity is not constant is illustrated by X = 5 – Y (a straight line of downward unity slope going through Y=0, X=5). This has:

$$S_Y^X = \frac{Y}{X}\frac{\partial X}{\partial Y} = \frac{-Y}{5-Y} \qquad (7)$$

It is going to blow up at Y=5. It is large (well -4) around Y=4 but back around Y=1 it is -1/4.

If we are not in the mood for partial derivatives, we can just estimate the sensitivity as:

$$S_Y^X = \frac{\frac{\Delta X}{X}}{\frac{\Delta Y}{Y}} \qquad (8)$$

For example, suppose we are designing with X = 5-Y in the vicinity of Y=1. We calculate

X(1) = 5 – 1 = 4

and then we jiggle Y a bit, say Y = 1.01 so that X = 3.99. Thus ΔY = +0.01 and ΔX = -0.01, so:

$$S_Y^X = {}^{\Delta X}/_{X}\Big/{}^{\Delta Y}/_{Y} = {}^{-0.01}/_{4}\Big/{}^{0.01}/_{1} = -\tfrac{1}{4} \qquad (9)$$

which in this case comes out exactly what we had with the partial derivative.

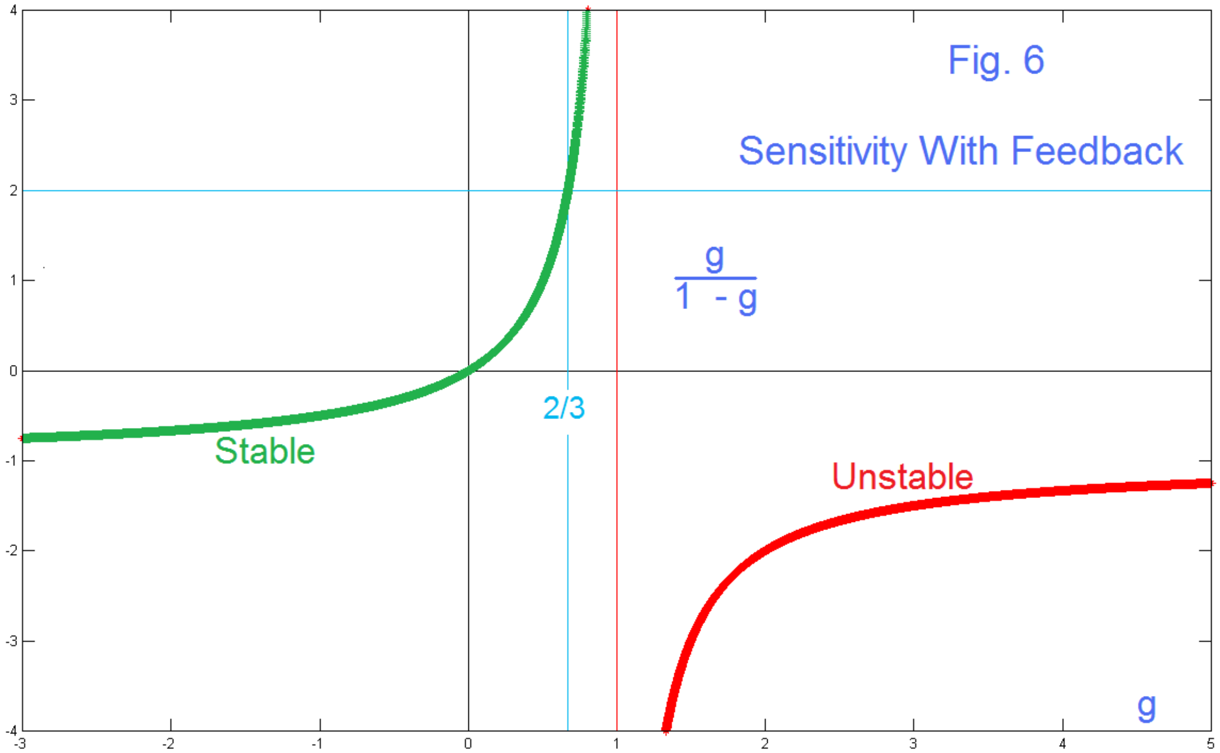
The notion of a tuning equation [2b, 2c] is likely clear at this point. We build something, an actual instance or a designed circuit and perhaps it is supposed to have a Q of 10 but we test it as having a Q of 9.3. We have a design equation of Q as related to some resistor R_Q (perhaps nominally 27k) and have calculated $S_{R_Q}^{Q}$ and evaluated this at Q=10, and suppose this came out at 2. How do we change R_Q?

$$\Delta R_Q = \left(\frac{R_Q}{Q}\right)\left[\frac{1}{S_{R_Q}^{Q}}\right]\Delta Q = \frac{27k}{10}\left(\frac{1}{2}\right)(+0.7) = 945\ ohms \qquad (10)$$

So, we would add a 910 ohm 5% resistor. Why is this better than a 27k 5% resistor? Well, the measurement of Q is effectively a measurement of R_Q, and the correction should be to the accuracy with which we measured Q.

APPLYING SENSITIVITY TO THE POSITIVE FEEDBACK GAIN

Above we solved for the gain G with feedback g, then went on to look at the general issue of classical sensitivity. While we generally thought of sensitivity with respect to some passive component, it can be done with respect to any parameter, including g.



Equation (4), just G = 1/(1-g) gives us G as a function of g, and thus the sensitivity of G to g is:

$$S_g^G = \frac{g}{G}\frac{\partial G}{\partial g} = \frac{g}{1-g} \qquad (11)$$

This is remarkably simple and easily confused with G itself (the sensitivity is gG), and is plotted in Fig. 6, which we need to compare carefully with Fig. 5. Note that at g=2/3, the sensitivity is 2 (while G=3). Is this correct

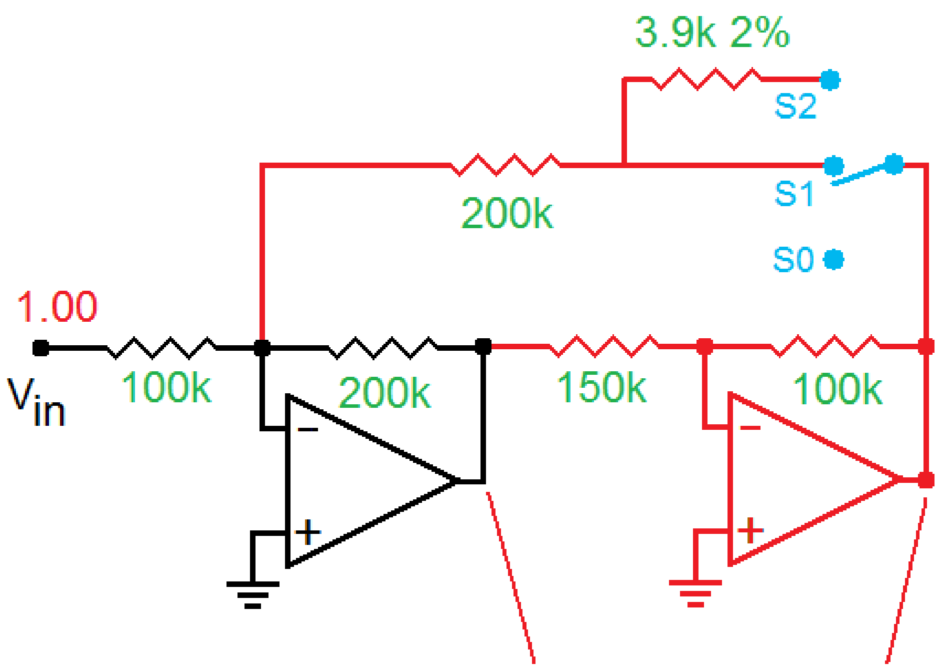
To illustrate the validity we can set up an experimental circuit, and this is shown in Fig. 7 where a circuit with the gain of 2 has a positive feedback around it with a gain of 2/3. This should give:

$$G = 2\,\frac{1}{1-2/3} = 6 \qquad (12)$$

Note the three-position switch, and the table that gives theoretical values and experimental observations (with experimental error to keep in miind). With the switch in S0 position, the feedback loop is broken, and the gain is just 2.   In the S1 position, the positives feedback of 2/3 is working, and indeed, the gain of 6 is now seen. The trick next is to switch in a small series resistor. We 3.9k as about a 2% addition to the red 200k resistor. (This is a perfectly valid ploy even though we have not accurately measured the 200k.). This decreases the feedback gain by about 2%. Note that the feedback gain is set by the ratio (100k/150k) at 2/3 which is then inverted and fed back by the ratio of the two 200k resistors. The 100k resistor in the input merely sets the open loop gain at 2.

The happy result is then that in the S2 position the gain drops by 4%. Exactly what a sensitivity of 2 should give (2 x 2%). Excellent agreement.

Fig. 7



S0	-1.97	1.32
S1 Theory	-6	4
S1	-6.16	4.17
S2	-5.92	4.01

4%

Experimental Circuit

REFERENCES

[1] B. Hutchins, "Feedback Revisited – Gain Due to Feedback", ***Electronotes***, Volume 23, Number 219 November 2013 http://electronotes.netfirms.com/EN219.pdf

[2a] B. Hutchins, "Analog Signal Processing, Chapter 7, Passive and Active Sensitivity", ***Electronotes,*** Volume 20, No. 195, July 2000 http://electronotes.netfirms.com/EN195.pdf

[2b] B. Hutchins, "Tuning Equations Derived from Passive Sensitivity", ***Electronotes,*** Volume 20, No. 196, Dec. 2000 http://electronotes.netfirms.com/EN196.pdf

[2c] B. Hutchins, "Passive Sensitivity and Tuning Equations," Electronotes Application Note AN-361, May 15, 2006 http://electronotes.netfirms.com/AN361.pdf

NOTES ON "WATTS UP WITH THAT" WEBSITE

The following three links from Anthony Watts' remarkable "Watts Up With That" website correspond to Parts 1- 3 (Aug 27, Sept 3, and Sept 6 of 2016) of (Lord) Christopher Monckton's multi-part post which supposedly deals with climate feedback. The postings include numerous comments, including some from engineers such as myself to inform Monckton that he does not understand feedback the way engineers use it, and he is factually wrong. In addition to being demonstrably wrong, while accusing his critics of rudeness, he is himself exceptionally rude. He insists that his critics should wait to attack his presentation on feedback until he posts later installments. What he has already posted is embarrassingly wrong and boorish. We wait for Part 4 (etc?) as of Sept 27, 2016 – perhaps hoping for an apology.

https://wattsupwiththat.com/2016/08/27/feet-of-clay-the-official-errors-that-exaggerated-global-warming/

https://wattsupwiththat.com/2016/09/03/feet-of-clay-the-official-errors-that-exaggerated-global-warming-part-2/

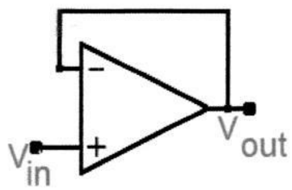
https://wattsupwiththat.com/2016/09/06/feet-of-clay-the-official-errors-that-exaggerated-global-warming-part-3/

Note Added March 2, 2017: To date Monckton has not posted further. Perhaps someone suggested that HIS LORDSHIP stop making a fool of himself. Two small typos also repaired this date.

Appendix A Real/Ideal

	IDEAL OP-AMP	REAL OP-AMP
Open Loop Gain	Infinite	Very large at DC (perhaps 10^7) but intentionally set to roll-off (for stability) at higher frequencies.
Input Bias Currents	0	very small – a few picoamps
Slew Rate of Output	Infinite	perhaps 10 volts/microsecond
Output Impedance	Zero	100 Ohms, reduced by negative feedback as used
Differential Input Offset	Zero	a few mV

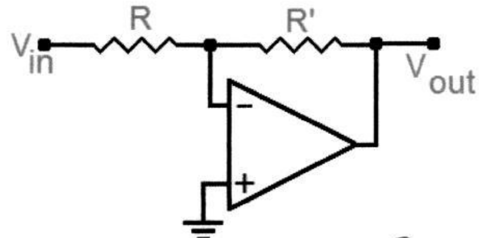
Appendix B   Op-amp Circuits - Hints



$V_{out} = V_- = V_+ = V_{in}$

$V_{out} = V_{in}$

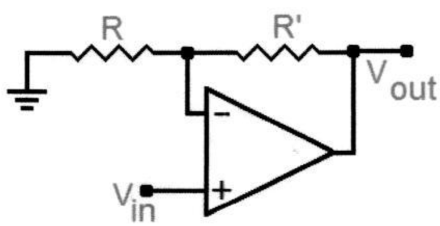
Follower



Appendix B

$V_- = \frac{V_{in} \cdot R' + V_{out} R}{R' + R} = 0 \qquad V_{out} = -\frac{R'}{R} V_{in}$

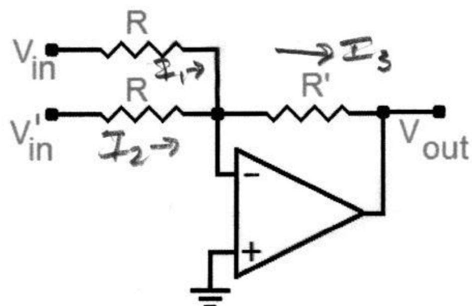
Inverter



$V_- = \frac{V_{out} R}{R + R'} = V_{in}$

$V_{out} = \frac{R + R'}{R} V_{in} = \left[1 + \frac{R'}{R}\right] V_{in}$

Non-Inverter



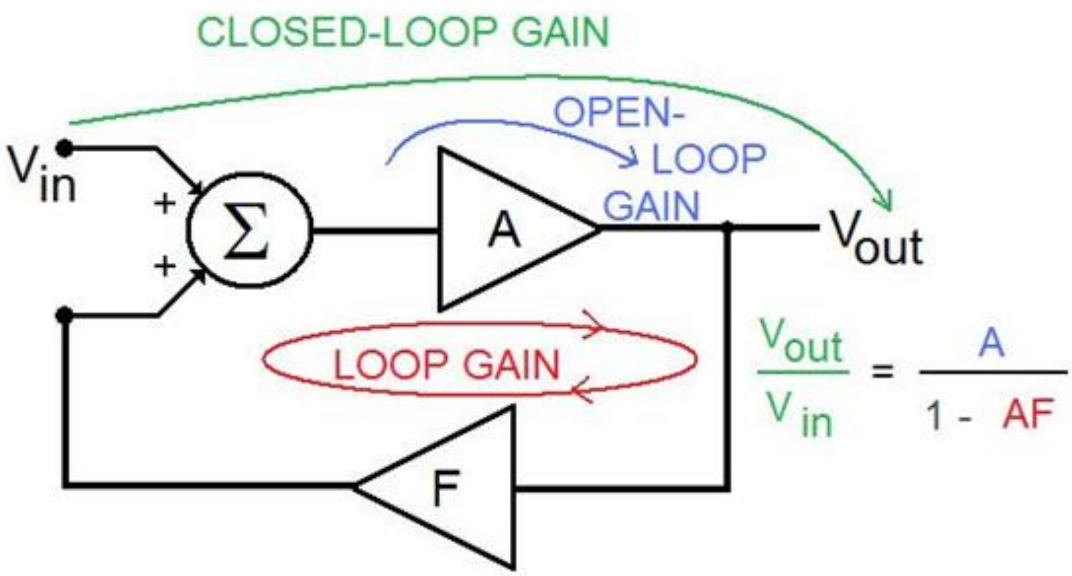
$I_1 + I_2 = I_3 \qquad V_- = V_+ = 0$

$\frac{V_{in}}{R} + \frac{V_{in}'}{R} = -\frac{V_{out}}{R'}$

$V_{out} = -(V_{in} + V_{in}')(R'/R)$

Inverting Summer / Amp

Appendix C Loop Gains



Here the feedback factor is F instead of β since the figure was created for a blog post that used that notation.   Again we emphasize the advisability of working out each presentation individually.

Appendix D Sensitivity

Classical Sensitivity Examples   Bernie H 8/30/2016

$$S_Y^X = \frac{\frac{\Delta X}{X}}{\frac{\Delta Y}{Y}} \approx \frac{Y}{X}\frac{\partial X}{\partial Y}$$

Appendix D

Suppose for some filter we have

$$Q = \frac{3R_G}{R}$$

$$S_{R_G}^{Q} = \frac{R_G}{Q}\frac{\partial}{\partial R_G}\left(\frac{3R_G}{R}\right) = \frac{R_G}{Q}\,\frac{3}{R} = \frac{R_G}{3R_G}R\,\frac{3}{R} = 1$$

% change in R_G → <u>same</u> % change in Q   <u>Good</u>

Suppose also the cutoff freq =

$$\omega_0 = \frac{1}{\sqrt{R_1R_2C_1C_2}}$$

$$S_{R_1}^{\omega_0} = \frac{R_1}{\omega_0}\frac{\partial \omega_0}{\partial R_1} = \frac{R_1}{\omega_0}\frac{\partial}{\partial R_1}\left[R_1R_2C_1C_2\right]^{-1/2}$$

$$= \frac{R_1}{\omega_0}\left[R_2C_1C_2\right]^{-1/2}\frac{\partial}{\partial R_1}\left(R_1^{-1/2}\right)$$

$$= R_1\left[R_1R_2C_1C_2\right]^{1/2}\left[R_2C_1C_2\right]^{1/2}\left(-\frac{1}{2}\right)R_1^{-3/2}$$

$$= -\frac{1}{2}$$ <u>better</u>

Or if $Q = \frac{1}{3-K}$ ("Sallen-Key")

$$S_K^Q = \frac{K}{Q}\frac{\partial}{\partial K}(3-K)^{-1} = \frac{K}{Q}(-1)(3-K)^{-2}(-1)$$

$$= \frac{K}{(3-K)^2}(3-K) = \frac{K}{3-K} = 3Q-1$$

<u>not good</u> as Sensitivity increases with Q

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