ELECTRONOTES
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CASCADING VOLTAGE DIVIDERS – AND FRIENDS

INTRODUCTION:

Possibly nothing is more familiar to us than the standard resistor voltage-divider (Fig. 1a). Indeed one need not wait for an electrical engineering class to encounter this – it being a primary example from beginning physics*. It is so familiar we use it without thinking, although at some point there were some essential assumptions involved. As shown, there is a single current (I) through both resistors, and the current is $V_{in}/(R_1+R_2)$.

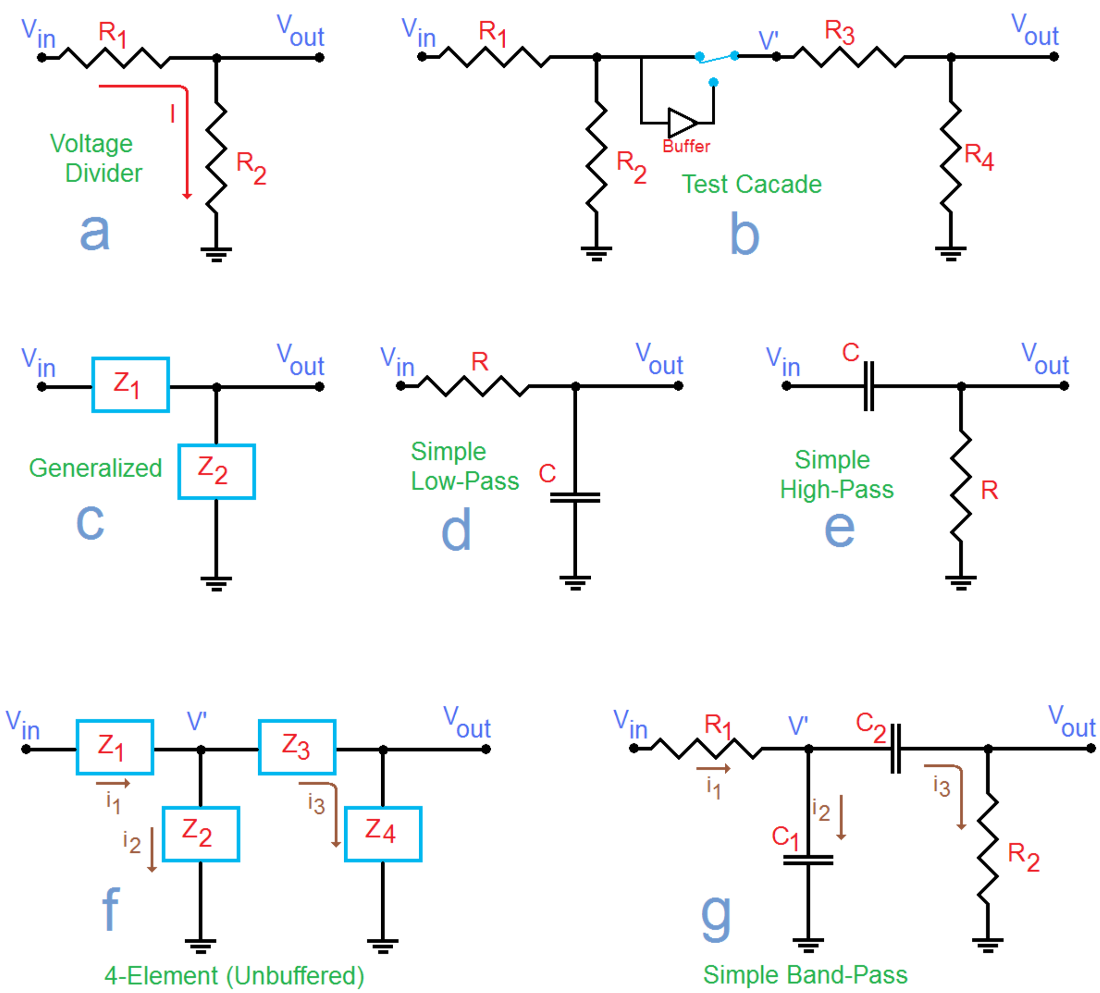


Fig. 1

The output voltage V_{out} is thus IR_2 or $V_{in}[\ R_2 / (R_1+R_2)\ ]$ and that's the story. Implicit here are two notions**. First, the voltage (source) has zero output impedance; else that impedance needs to be added to R_1 and becomes part of the calculation. Secondly, and important here, the voltage V_{out} (output) must be measured by a very high impedance, or passed on by something like the input stage of an op-amp which draws only a tiny current. We do not want to steal any portion of the standing current I. So we have two voltages in the diagram, V_{in} and V_{out} which are fundamentally different. The source V_{in} is an imposed voltage (a mathematical known) with zero source impedance (at least in theory) while the output V_{out} is a voltage to be used later (a mathematical unknown), but not as anything capable of supplying significant current. Too often, we automatically make allowances for inputs vs. outputs, without explaining.

Given that we can't significantly "load" the output of a voltage divider without making perhaps significant allowances, we can't expect to be able to cascade either voltage dividers, or more general structures (see below) without changing what happens at the first junction. Any second stage "loads" the first. Well, we could cascade if we use a "buffer" which is shown in Fig. 1b as one of the switched selections (not selected in the figure). That is, the buffer (usually an op-amp voltage-follower) monitors the input and supplies a low impedance re-sourcing of its input voltage. With the buffer, the cascade works and the total attenuation is the product of the attenuations of the individual stages. Without the buffer, the second stage will always load down the first, increasing the attenuation to some degree.

We should remark that we are mainly interested in this cascading when the stages are simple low-pass and high-pass filters (see below) and in general, there is little likelihood or a need for cascaded resistive attenuators. A cascade may be part of a control structures (such as a mixing board). Or a load equivalent to a second stage may be the result of a variable voltage divider (a potentiometer!) feeding a realistic load. And, some extremes requiring very large attenuation ratios do occur***.

One simple approach which would work to some degree is to make the impedance of the second stage much higher than that of the first, while maintaining the desired attenuation ratio. For example, suppose in Fig. 1b that (without the buffer) R_1 is 10k and R_2 is 1k (an attenuation of 1/11 = 0.9090…). If we followed this by the same attenuator for R_3 and R_4, the additional attenuation would be somewhat greater. If however, we used R_3=1M and R_4=100k, the attenuation of the second stage would be 1/11 as before, and the 1.1M resistance of the second stage would have little effect on R_2. (The difference would be that the output impedance of V_{out} would be much higher, nearly 90k.) But we really do not expect this to come up.

More to the point, let's consider the case where these are not resistive voltage dividers, but where capacitors are involved as simple low-pass and high-pass filters. Fig. 1c shows the divider composed of two unspecified impedances Z_1 and Z_2. If the impedance is a resistance, we write Z=R. If it is an inductor, we write Z = sL and if it is a capacitance, we write Z = 1/sC. Here we will only consider capacitors. Here s is the familiar complex Laplace variable. In these cases where the impedances are more general (meaning frequency dependent) we describe the attenuation not as a constant but as a <u>frequency response</u>, the <u>magnitude</u> of a "transfer function" which has the notation $T(s) = V_{out}(s)/V_{in}(s)$. Recall that the mathematical path from T(s) to the frequency response, denoted FR= |T(jω)| = |T(2πf)|, is to substitute jω = 2jπf for s. As with any complex number, you then multiply T(jω) by T(-jω), it's complex conjugate, and take the square root. This is nothing more than the Pythagorean Theorem.

The transfer function corresponding to [R_2 / (R_1+R_2)] is for the general case of Fig. 1c:

$$T(s) = \ [\, Z_2 \,/\, (Z_1 + Z_2)] \tag{1}$$

For the low-pass of Fig. 1d this becomes:

$$T_L(s) = \ [\, 1/\,(1 + sCR)] \tag{2}$$

and for the high-pass of Fig. 1e it becomes:

$$T_H(s) = \ [\, sCR/(1 + sCR)] \tag{3}$$

equations (2) and (3) are a roll-off and a roll-up of 6db/octave with 3db cutoff ($\frac{1}{\sqrt{2}}$) of $1/2\pi RC$. Nothing new here.

We argued above that there was little or no reason to ever cascade voltage dividers. On the other hand, with the cases of the filters we can consider cascading for different filters and/or increased roll-off rates. This approach may be a dead end. First, there is the loading problem discussed above. Secondly, and probably far more importantly, we can obtain a greater roll-off rate (asymptotically), but no respectable (sharp) corners****. The flaw is that the network is passive. In order to achieve sharp corners we need complex poles. One way to achieve complex poles is to use inductors (coils), which can be large and heavy (iron cores) at audio frequencies. Accordingly, "active filters" which use a power supply and op-amps are generally the choice [1,2].

So – lots of not so good ideas here. But there remains a simple case that yields an interesting result. This is suggested as a general 4-element cascade (Fig. 1f) with a specific choice of a cascade of a low-pass and a high-pass (Fig. 1g) to achieve a rudimentary band-pass [3].

We can either solve the general network of Fig. 1f and plug in R's and C's, or we can solve Fig. 1g directly [3 – third link], both leading to the same answer. In these cases, note that we have the known source voltage, and now <u>two unknown voltages</u>, V_{out}, and the intermediate node voltage called V' here. It is still true that V_{out} is related to V' as (using the general approach):

$$V_{out} = \frac{V' Z_4}{Z_3 + Z_4} \tag{4}$$

which is one of the two equations (two unknowns) we need. The second equation is obtained by summing currents at the V' node:

$$i_1 = i_2 + i_3 \tag{5}$$

where:

$$i_1 = \frac{V_{in} - V'}{Z_1} \tag{6a}$$

$$i_2 = \frac{V'}{Z_2} \tag{6b}$$

$$i_3 = \frac{V'}{Z_3 + Z_4} \tag{6c}$$

Plugging equations (6) into equation (5) and eliminating V' in favor of V_{out} using equation (4) yields:

$$T(s) = \frac{V_{out}}{V_{in}} = \frac{Z_2 Z_4}{[\, Z_1 Z_2 + (Z_3 + Z_4)(Z_1 + Z_2)\,]} \tag{7}$$

Next, plugging in the elements of Fig. 1g corresponding to Fig. 1f, we have:

$$T(s) = \frac{V_{out}}{V_{in}} = \frac{\dfrac{R_2}{sC_1}}{\left[\dfrac{R_1}{sC_1} + \left(\dfrac{1}{sC_2} + R_2\right)\left(R_1 + \dfrac{1}{sC_1}\right)\right]} \tag{8a}$$

$$= \frac{\dfrac{s}{R_1 C_1}}{\left[\, s^2 + s\left(\dfrac{1}{R_2 C_1} + \dfrac{1}{R_2 C_2} + \dfrac{1}{R_1 C_1}\right) + \dfrac{1}{R_1 C_1 R_2 C_2}\right]} \tag{8b}$$

Having calculated T(s) as in equation (8b) is the key to everything else: poles/zeros, frequency response, phase response, etc. [1]. In the manner outlined above, we could write a closed form for the frequency response |T(s)|. However, we can also just use a program function such as Matlab's ***freqs*** directly on the polynomial coefficients of equation (8b). So here, we will cut a few corners. We know that the low-pass, by itself would have a pole at $-1/R_1C_1$; and the high-pass, by itself, would have a pole at $-1/R_2C_2$. If we use a buffer between the stages, or here, just make sure the impedance level of the high-pass is significantly larger than that of the low-pass, we might just suppose the poles were uncoupled. What would the denominator look like in such a case?

$$\left(s-\left(-\frac{1}{R_1C_1}\right)\right)\left(s-\left(-\frac{1}{R_2C_2}\right)\right) = \left[s^2 \ + s\left(\frac{1}{R_2C_2}+\frac{1}{R_1C_1}\right)+\frac{1}{R_1C_1R_2C_2}\right] \tag{9}$$

which is the same as the denominator of equation (8b) except for the addition to the damping term (coefficient of s) of $\frac{1}{R_2C_1}$ for equation (8b). Thus the poles were assumed purely real, and they are moved even further from ever becoming complex as a result of the coupling. We expected this. The "Q" (sharpness) of the band-pass (the reciprocal of the damping) is very low. (If we DO need the actual poles <u>exactly</u>, we leave the $\frac{1}{R_2C_1}$ term in the denominator and use the quadratic formula.)

Using the ***freqs*** function (or similar, or functionally equivalent) we can easily calculate the frequency response. The function ***freqs*** requires the three polynomial coefficients of the numerator [0, $\frac{1}{R_1C_1}$, 0] and of the three polynomial coefficients of the denominator [1 , $\frac{1}{R_2C_1}+\frac{1}{R_2C_2}+\frac{1}{R_1C_1}$, $\frac{1}{R_1C_1R_2C_2}$] as well as a frequency vector: <u>samples</u> of $\omega = 2\pi f$, where f is ordinary frequency in Hz. For example, if we want to calculate on the interval 0 to 50 Hz, we might choose ω = 2π [0 0.1 0.2 0.3 …… 49.9 50.0]. For the frequency response magnitude, we just take the absolute value of ***freqs***.

Here we shall show three example filters, two from the spec. sheet [3 – 2nd link] and one as an example of a narrower response with higher cutoffs. The frequency responses are shown in Fig. 2, corresponding to the data in the table below the figure. We see the expected very low-Q band-pass responses with 3db frequencies very close to the design values. We have not discussed how high these responses should be at maximum, but it is clear that they approach but do not reach 1, and that the narrower response (dashed black) is lower than the other two, and this is as expected. It is well-known [4] that the center, max of the band-pass, is the square root of the product of any two points at the same level (traditionally the -3db = 0.707, green line, is used). This center frequency is named ω_0, and $\omega_0^2 = \frac{1}{R_1C_1R_2C_2}$ is the third term of the denominator polynomial of equation (8b).

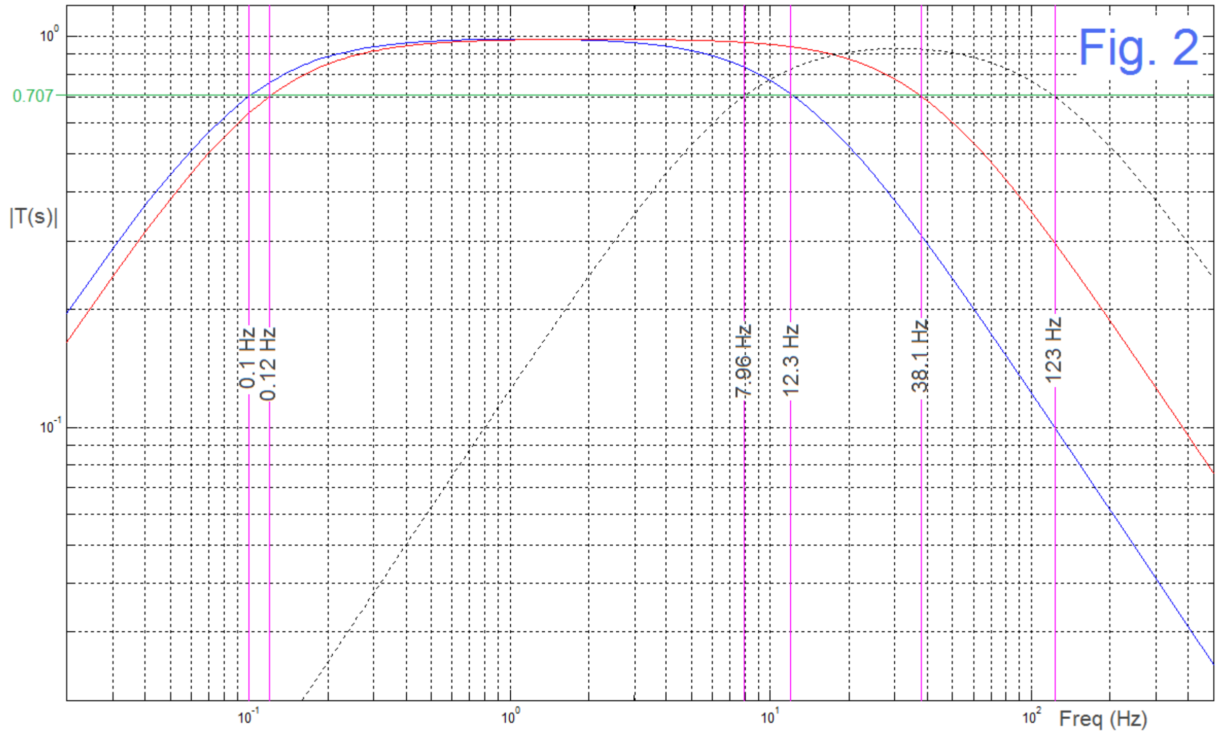


TABLE OF EXAMPLE FILTERS (resistors in kΩ, capacitors in μF)

Color	R_1	C_1	R_2	C_2	HP=$1/2\pi R_2C_2$	LP =$1/2\pi R_1C_1$	max(|T(s)|)
Blue	1.9	6.8	233	6.8	0.10 Hz	12.3 Hz	0.984
Red	1.9	2.2	133	10	0.12 Hz	38.1 Hz	0.983
Dashed Black	1.9	0.68	200	0.1	7.96 Hz	123 Hz	0.931

Now notice that if we evaluate equation (8b) at s=jω_0, the s^2 term becomes $\frac{-1}{R_1C_1R_2C_2}$ so this cancels the constant term (ω_0), and then the s in the numerator cancels the s in the denominator. This leaves a constant, the magnitude of T(s) at ω_0, the peak gain:

$$\max(|T(s)|) = T(j\omega_0) = \frac{R_2C_2}{[R_1C_2+R_1C_1+R_2C_2]} \qquad (10)$$

As long as we arrange $R_1C_2 + R_1C_1 \ll R_2C_2$, the case here, this peak (plateau) can be sufficiently close to 1 for most purposes.

Matlab Code for Fig. 2

```
% AN429.m
%
%Filter 1
R1=1900
C1=6.8e-6
R2=233000
C2=6.8e-6
P1=1/(2*pi*R1*C1)
P2=1/(2*pi*R2*C2)
%
n1=1/(R1*C1)
d2=1
d1=(1/(R1*C1) + 1/(R2*C2) + 1/(R2*C1))
d0=(1/(R1*C1*R2*C2))
%
f=[0.01:.01:20];
f=[f [20.1:0.1:500]];
w=2*pi*f;
%
T1=abs(freqs([0 n1 0],[d2 d1 d0],w));
T1max = max(T1)
T1Calc=R2*C2/(R1*C2+R1*C1+R2*C2)
%
% Filter 2
R1=1900
C1=2.2e-6
R2=133000
C2=10e-6
P1=1/(2*pi*R1*C1)
P2=1/(2*pi*R2*C2)
%
n1=1/(R1*C1)
d2=1
d1=(1/(R1*C1) + 1/(R2*C2) + 1/(R2*C1))
d0=(1/(R1*C1*R2*C2))
%
T2=abs(freqs([0 n1 0],[d2 d1 d0],w));
T2max = max(T2)
T2Calc=R2*C2/(R1*C2+R1*C1+R2*C2)
%
% Filter 3
R1=1900
C1=0.68e-6
R2=200000
C2=0.1e-6
P1=1/(2*pi*R1*C1)
P2=1/(2*pi*R2*C2)
%
n1=1/(R1*C1)
d2=1
d1=(1/(R1*C1) + 1/(R2*C2) + 1/(R2*C1))
d0=(1/(R1*C1*R2*C2))
%
```

```
T3=abs(freqs([0 n1 0],[d2 d1 d0],w));
T3max = max(T3)
T3Calc=R2*C2/(R1*C2+R1*C1+R2*C2)
%
%
%
figure(1)
loglog(f,T1)
hold on
loglog(f,T2,'r')
loglog(f,T3,'k:')
plot([0.001 500],[1/sqrt(2) 1/sqrt(2)], 'g')
   plot([12 12],[0.001 2],'m')
   plot([38 38],[0.001 2],'m')
   plot([.1 .1],[0.001 2],'m')
   plot([.12 .12],[0.001 2],'m')
   plot([P1 P1],[0.002 2],'m')
   plot([P2 P2],[0.002 2],'m')
hold off
grid on
axis([.02 500 0.02  1.2])
```

REFERENCES

[1] The basic ideas of analog signal processing were covered in a comprehensive series of ***Electronotes***, and in numerous places in our publications. Here we note in particular "Analog Signal Processing", Sections 1-5 ("Transfer Functions by Network Analysis") and Section 1-6 ("Frequency response, Poles and Zeros, Impulse Reponses") **Electronotes** ,Vol. 19, No. 191 Dec. 1999 http://electronotes.netfirms.com/EN191.pdf

The full series runs from Issue 191 to Issue 196.

[2] "Looking Again at the RC Low-Pass," ***Electronotes*** Volume 22, Number 210 May 2012 http://electronotes.netfirms.com/EN210.pdf

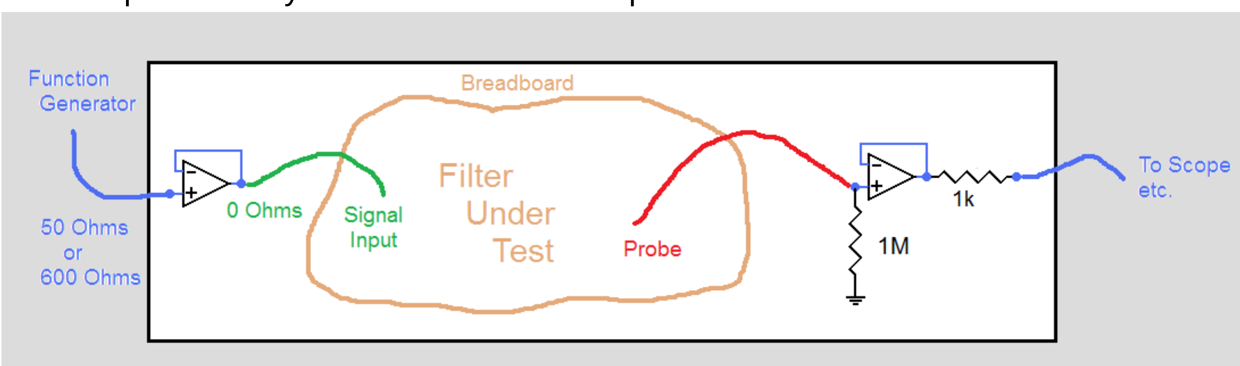
[3] See Glen's website at https://hummap.wordpress.com/2016/07/16/just-before-i-go-off-the-grid-for-a-few-weeks/ where he links to http://volcanomodels.sr.unh.edu/jbj/MICROPHONES/J1%20Basic.pdf . See also my response that led to this AN http://electronotes.netfirms.com/micfilter.pdf

[4] "Low-Q Bandpass Filter," Electronotes App. Note No. 25, Feb. 4, 1977, and "Frequency Response of Bandpass Filter," Electronotes App. Note No. 271, Feb 15, 1983 http://electronotes.netfirms.com/AN271.PDF

Historical Notes Just for Fun

* I am sure this is taught in a beginning physics course in college – perhaps also in high school. Anyway I knew it in HS through self-study and from explaining electronics to my friend, classmate, (and first student!) Jerry DeGraff. I explained the voltage divider, which he understood, and I had already explained to him how a triode vacuum tube worked to amplify a voltage. Prefacing a query with the polite qualification that it was "probably a silly question", Jerry asked why, given that I was so intent on amplifying a voltage that I was now intending to make it smaller. I never forgot his question, and the need, from time to time, to say why we are doing the things we are. Too many students just don't ask. In more recent time, I would teach "Perfect Reconstruction Filters" (breaking a signal into parts in an unlikely way – and putting it back together). I always had to prompt the students to ask why we bothered. It was because we could monkey with the pieces, compressing data, before less perfect (but acceptable) reassembly.

** Attaching an instrument may significantly disturb a proper measurement. In our student labs we often saw errors (by students AND instructors) where allowance for the output impedances of sources, or the capacitive loading of scope cables, was not recognized. If testing a first-order LP RC such as we have seen here, we might have a resistor R= 3k and a function generator adding 600 ohms to it, and find folks wondering why they were off by 20%. In a serious lab we more or less had two op-amp followers installed permanently on either end of our experimental breadboards.



The left op-amp buffered the function generator, offering essentially 0 ohms of output impedance. The right op-amp isolated the scope cable from any op-amp outputs with the 1k series resistance (little effect on the scope display, but no unstable oscillations). It also offered a 1M "probe" to hold down the voltage when the probe wire (red) was just floating or being moved. Trying to connect cables with alligator clips usually meant frustration with components pulled out and constant cable repair. With our setups, we generally just moved the green and red wires to different breadboard holes as needed. Sure the students needed to learn in lab that the real world was tricky. But it was a good idea to avoid known pitfalls. There were enough unknown unknowns automatically.

*** People tend to suppose that there are some limits to resistive voltage dividers – some maximum attenuation beyond which any additional cut never is needed. Perhaps 1000:1 or something like that. The idea is related to the notion that if you cut a signal too much, it is down in the inherent noise/offset regions.   That is, to a millivolt level; down from levels of a few volts.

But what if your signals (voltage levels) were thousands of volts, and you proposed to regulate this level with an op-amp loop.   That's silly! Op-amps can't stand thousands of volts. Some 40+ years ago, my friend and fellow student Jim Starbuck, wanted to regulate a high-voltage lab supply of some 2000 volts DC (or was it 5000 volts – not sure!), and he proposed to do this with an IC op-amp (a 741 – new in those days). He proposed a triac in the AC line, with the op-amp controlling the triac. Then a feedback loop from the lab supply output to the op-amp input.   But how in the world do you feed 2000 volts to an op-amp? With a voltage divider of course. The op-amp only sees the divider output. Yes – it was wise to make the upper leg of the divider from several series resistors to avoid the possibility of arcing.

**** In my day, every EE student feared going home for vacation and being asked (told) to fix the family TV. Usually you could plead a lack of the right equipment. A very incomplete explanation! You didn't <u>know</u> how to fix a TV and never would. In my case, it was more the neighbor's senior play recording.   The play, late 50's, had been audio recorded with a poorly grounded microphone and was mostly hum. We all knew this was 60 cps (now Hz). I knew just enough to suppose that a high-pass filter could pass the speech while blocking 60 Hz.   I even knew how to make an RC high-pass with cutoff above 60 Hz. I really don't remember if I had any idea about impedances, but we played the original tape and made a new copy. It was no better!   I even remember trying two HP stages, and this didn't work either.

Things are usually not as simple as we first suppose, and in this case, there were so many things that I might have done wrong that I can't be sure at all. Later I learned that one fundamental "error" was that "60 Hz hum" is really 120 Hz second harmonic. That was likely the main problem. Another thing was that I had little idea that filters had roll-off rates that mattered, and that by using the cascade, the corner kept getting worse. It is unlikely I had a notion of one stage loading the next and thus altering the performance.

Eventually I learned about notch filtering, comb filtering, and even adaptive filtering. By the time I salvaged the play, no one really cared! And by the way, the hum was saturating the tape, so that portions of the speech near zero crossings could be heard. Somehow the ear/brain made some sense of it. Here is the reference.

http://electronotes.netfirms.com/AES5.PDF