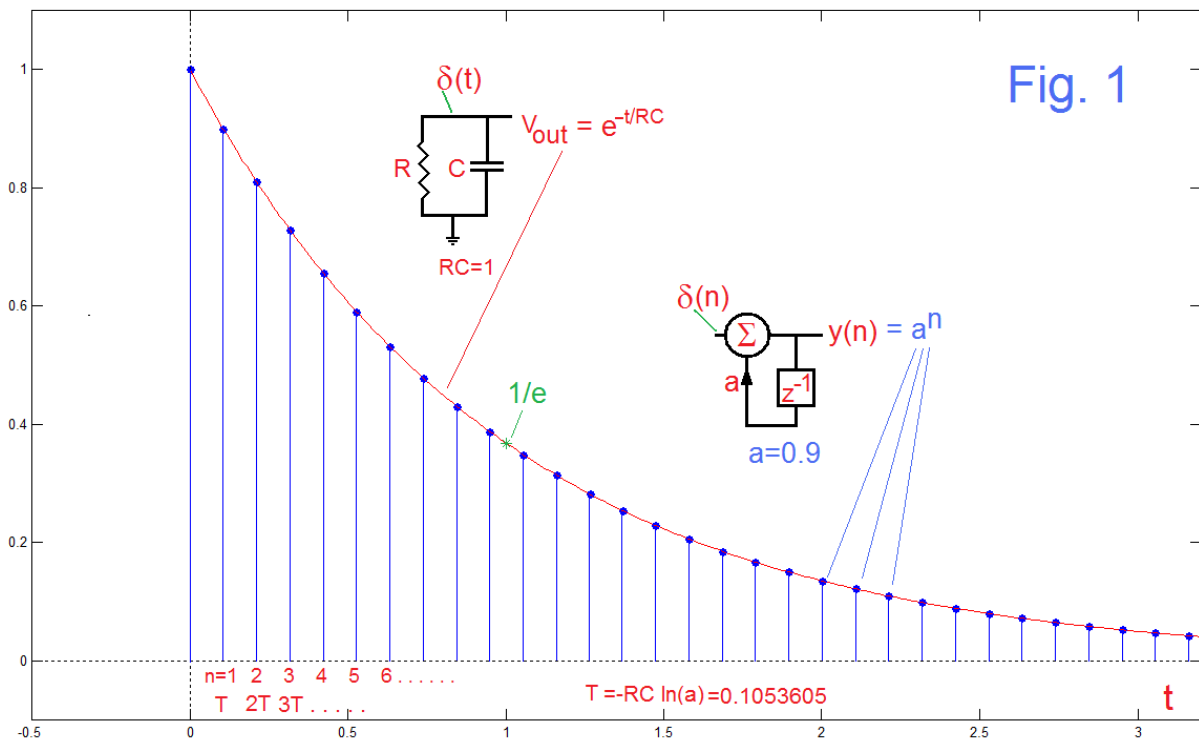


EXPONENTIAL DECAY – CONTINUOUS AND DISCRETE

IMPULSE RESPONSE:

Not unlikely one of the first “circuits” an engineering student encounters is the first-order R-C low-pass. Probably this is first solved by setting up the differential equation and solving it. Not unlikely this might have even been in a math class to suggest that calculus might be used for something real. The approach was perhaps disconcerting because it seemed to suggest that you first “guess” the answer and then plug in boundary conditions. Not the least of the worries, for the student peeking well ahead, was how this would be possible once a circuit with many more elements was the subject. At that point, no one said: “Don’t worry – You will eventually learn about Laplace Transforms.”



Most readers of these notes know very well that the first-order R-C (as in Fig. 1) has the discharge curve (red) as shown. You even know how to solve it (outlined here):

$$i = \frac{V_{out}}{R} = - \frac{dq}{dt} = -C \frac{dV_{out}}{dt} \tag{1}$$

for which we assume an exponential solution:

$$V_{out}(t) = Ae^{-t/\tau} \quad (2)$$

Plugging in this trial solution into equation (1) we find $\tau = 1/RC$ and then using the boundary condition $V_{out}(0) = 1$ we get $A = 1$ and thus:

$$V_{out}(t) = e^{-t/RC} \quad (3)$$

which is familiar. Exponential decay – the red curve. Note the green star at $t=1$ with value $1/e$.

Another actually simpler illustration of an exponential decay is shown in Fig. 1 by the discrete time system. Here the new output is found by taking the current output and multiplying by a , and repeating. For this example, taking $a = 0.9$, if we start at $n=0$ with $y(n)=1$, successive samples at the output are 1, 0.9, 0.81, 0.729, 0.6561,.... and so on. This discrete sequence is often called a “geometric progression” or “geometric series”, but it’s a sampled exponential function as can be seen. The blue “lollypop” samples are identical to what is on the red curve. We all know, more or less, that the geometric series and the exponential are related, but the conversion always seems to evade us momentarily. If the reader considers anything notable, it is how we actually got them to come out the same.

Here the key is to recognize from equation (3) that the only unique thing there is the RC time constant. Finding the discrete equivalent is a matter of choosing the correct sampling interval T for a desired geometric ratio a . This we see immediately if we write nT for t in equation (3):

$$V_{out}(t = nT) = [e^{-T/RC}]^n = a^n \quad (4)$$

so that:

$$e^{-T/RC} = a \quad (5a)$$

$$T = -\ln(a) RC \quad (5b)$$

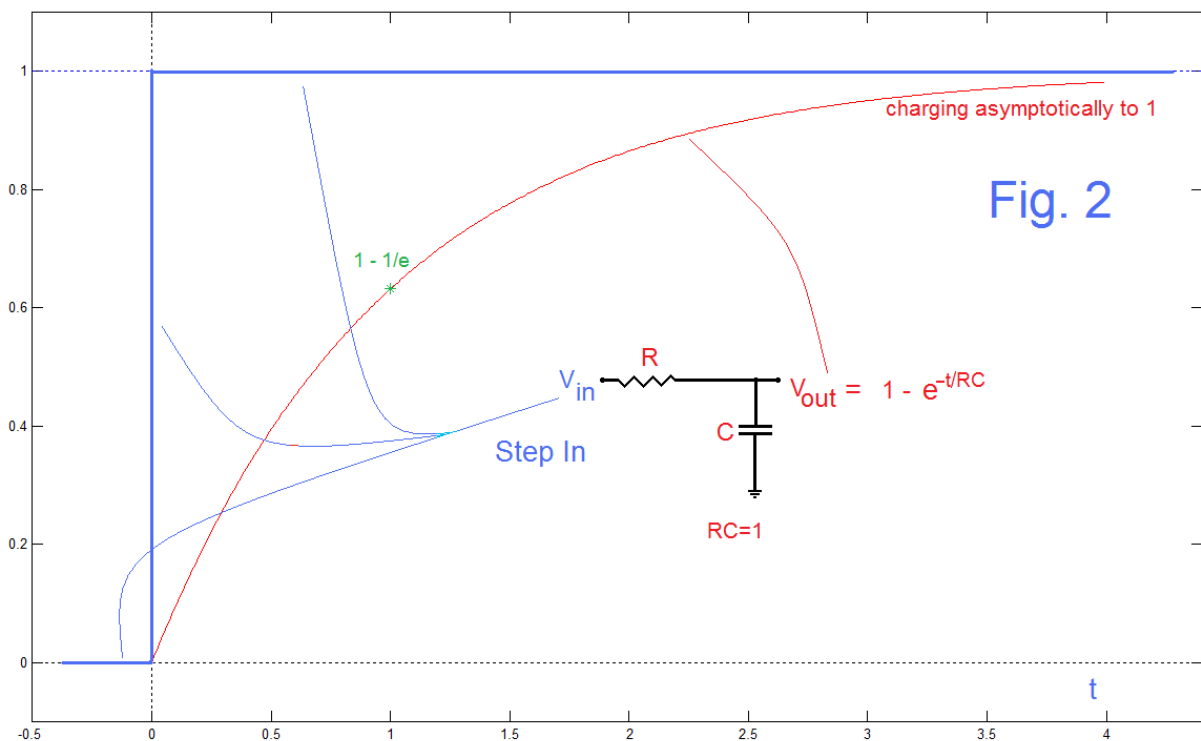
Thus in our example where $a = 0.9$, we have $T=0.1053605RC$. In Fig. 1, note that the $n=10$ (the eleventh) sample is at 1.013605, just beyond the $1/e$ point.

We thus see a significant relationship between the continuous and the discrete cases – at least for what we have displayed, which we would describe as the “impulse responses”. Thus Fig. 1 shows the imposition of a proper impulse as $\delta(t)$, the “Dirac Delta” for the continuous-time case, or $\delta(n)$, “Kronecker Delta” or “Unit Sample” in the discrete time case. These may not be essential because to the extent that we are not concerned with what was going on prior to $t=0$ (or $n=0$) we only need to worry about the function value at time 0, and the fact that there is no “input” thereafter (thus natural or unforced behavior). Next we need to consider what happens for different input conditions. For example, the “step response” is closely related to the impulse response, inverted.

STEP RESPONSE

While the notion of an impulse response is familiar, our best intuitive feel for a system is often in terms of its “step response”. In this case the RC circuit is not instantaneously charged by a Dirac delta, but the capacitor is initially uncharged and the resistor is connected not to ground, but to a voltage source. We get the familiar RC charge as in Fig. 2.

This is often misunderstood. We recognize that after enough RC time periods the charge is very close to the step size (it’s above 98% at $t=4$ in Fig. 2). This is a perfectly good way to actually charge a real RC circuit. Then grounding V_{in} would produce an exponential discharge as in Fig. 1. [Indeed, this is the usual way it is measured.] Readers of these notes will recognize the combined responses of Fig. 2 and Fig. 1 as the standard “AR” or attack-release envelope generator as used in music synthesizers. We can all sketch these curves, and probably put numbers on them. We easily remember that a charge or attack (Fig. 2) is a “convex” exponential while a discharge or decays is a “concave” exponential. **WRONG!**



The exponential response behaves in only ONE WAY, which is essentially concave. Presented with a change of input level, the output begins to change rapidly, and then slows (Fig. 1). In Fig. 2, the system is presented with a change of level, not from a positive level to zero, but from zero to a positive level. It really is just a subtraction from 1 for this case (a “discharge” to 1). In a general case, the output moves rapidly in the direction of change, and then slows (asymptotic).

LAPLACE TRANSFORMS

Engineers value the Laplace transform method as a way of deriving various time responses from a (frequency domain) transfer function. Laplace transform pairs are readily tabulated. Two that are of direct use here are:

$$\frac{1}{s-\beta} \leftrightarrow e^{\beta t} \quad (6a)$$

and:

$$\frac{1}{s(s-\beta)} \leftrightarrow \frac{e^{\beta t} - 1}{\beta} \quad (6b)$$

The transfer function of the RC low-pass in Fig. 2 is:

$$T(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} \quad (7a)$$

Thus the step response has Laplace transform:

$$T(s)/s = \frac{\frac{1}{RC}}{s(s + \frac{1}{RC})} \quad (7b)$$

Here with $\beta = -1/RC$ we verify the impulse response and the step responses as:

$$g(t) = e^{-t/RC} \quad \text{Impulse (Fig.1)} \quad (8a)$$

and:

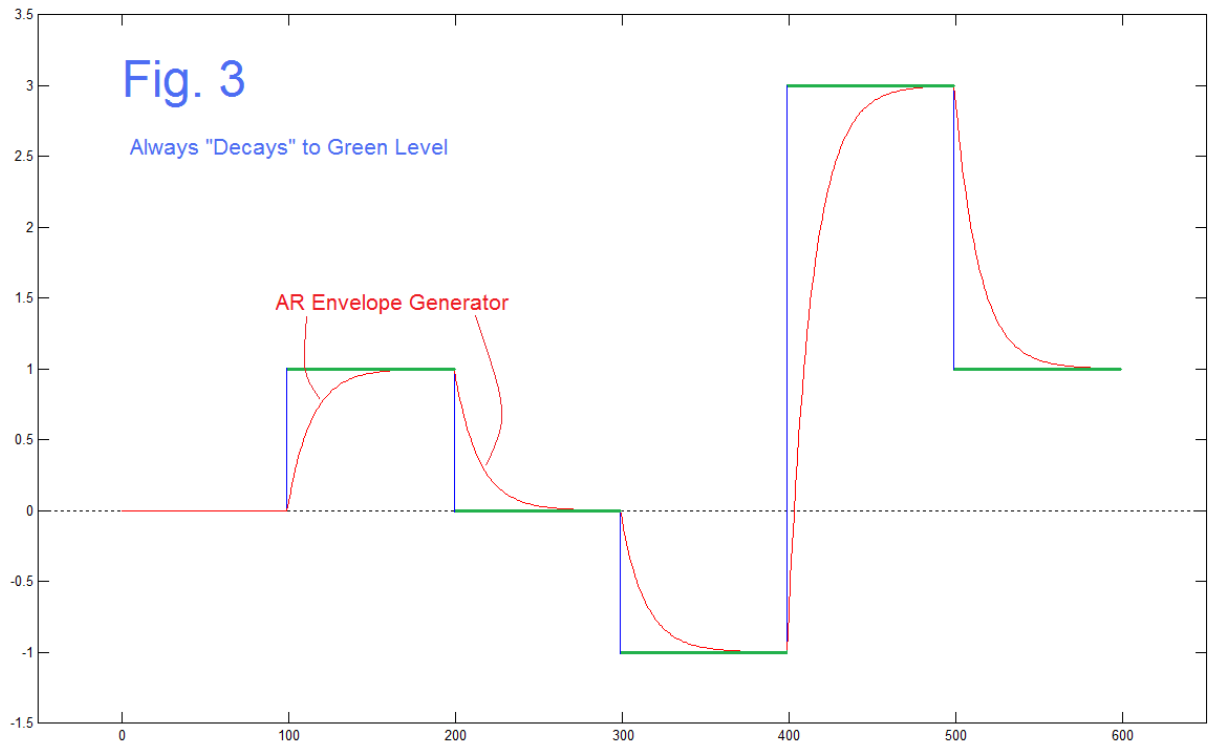
$$s(t) = 1 - e^{-t/RC} \quad \text{Step (Fig.2)} \quad (8b)$$

This verifies the previous results by the Laplace transform method – bypassing the need to solve differential equations

SIMULATION

Here we want to make the point about time response being a discharge to a level of a step change. This we could attempt with either differential equation solutions or Laplace transforms, but here the easiest thing is to simulate (numerically calculate), as this offers us a very general approach while giving up closed-form equations. A typical code line (here in Matlab) would be:

$$\mathbf{vout(n)=vout(n-1)+dt*(x(n)-vout(n-1))/(R*C);}$$



where the input (steps) are a sequence $x(n)$ and dt is the time increment (very small relative to RC). Fig. 3 is a typical example output. There are five target steps. The first two ($n=100$ to $n=200$) correspond to a usual step to a level of 1 such as Fig. 2. From $n=200$ to $n=300$ we have a discharge to 0, such as Fig. 1. In total, the range $n=100$ to $n=300$ is an example of an AR envelope generator used in music synthesizers. Further, it illustrates that we can observe the “impulse response” by charging the capacitor to an initial value very close to 1 with a step.

The remaining step changes from 0 to -1, from -1 to +3 (crossing zero), and from +3 to +1 (stopping short of zero) make the main point. In Fig. 3, the green step levels show the targets, and discharge (whether appearing as a charge or a discharge) are always to these green levels. Note that we have made all the steps of time length 100, but they could have been shorter or longer. The red curves are all the same function – just scaled and shifted. Note however that theoretically no red curve ever actually reaches the green steps. While we do not show it here, we could have had output curves falling noticeably short of the green targets. For example, a step length of 25 rather than 100 would have given about 78% at the top (here RC was 0.15 and dt was 0.01).

THE DISCRETE-TIME CASE

We have many times looked at results similar to those above for continuous-time [1,2]. While equally well established, the z-Transform methods for discrete-time, which we need to use here, are likely deserving of a reasonable outline.

Here we are dealing with the z-transform (essentially the Laplace transform applied to a discrete-time signal). For a discrete-time signal $x(n)$ the z-transform is:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (9)$$

so for a discrete step $s(n) = 1$ for $n \geq 0$ and $s(n)=0$ for $n<0$, we have [3]:

$$S(z) = \frac{1}{1-z^{-1}} \quad (10)$$

We easily write down the transfer function of the discrete network inset in Fig. 1:

$$H(z) = \frac{1}{1-az^{-1}} \quad (11)$$

We note from equation (11) that when we set $z=1$, the “DC” response, we get a gain of $1/(1-a)$, NOT a gain of 1 as we had for the DC response of the R-C circuit. Accordingly we expect the step response of $H(z)$ to go to $1/(1-a)$, which would be to 10 for the example of Fig. 1. So if we were going to use a step to “charge” the discrete time network, we would (strictly speaking) need to include an attenuation of $(1-a)$. To see what the whole step response is (not just the far end), as in equation (7b) for the R-C circuit, we multiply the transfer function by the transform of the step, and take the inverse transform of the product.

For the discrete-time case, we thus have:

$$Y_s(z) = S(z)H(z) = \frac{1}{1-z^{-1}} \frac{1}{1-az^{-1}} = \frac{1/(1-a)}{1-z^{-1}} + \frac{-a/(1-a)}{1-az^{-1}} \quad (12)$$

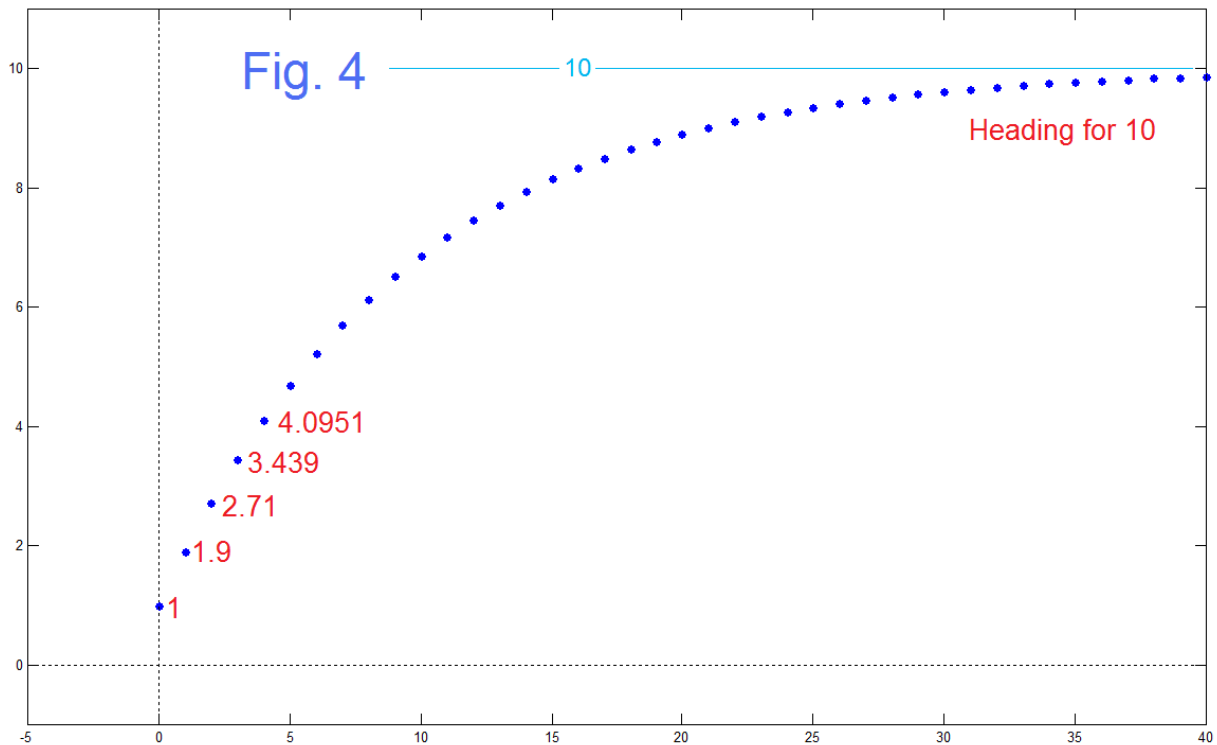
where the last term in equation (12) is the partial fraction version of the product, which we need to do the inverse z-transform by inspection (from first-order terms).

$$y_s(n) = \left[\frac{1}{1-a} \right] [s(n) - a a^n] = \left[\frac{1}{1-a} \right] [s(n) - a^{n+1}] \quad (13)$$

This is exactly the correct answer (Fig. 4).

We are aware that we could find the same sequence as is plotted in Fig. 4 by just simulating the network. A sample equal to 1 arrives at the input and it becomes the first output. That output is multiplied by 0.9 and as added to a new input sample (=1) and we get 1.9 as a second output. Then 1.9 is multiplied by 0.9 giving 1.71 which is added to the new input (still = 1) and we get 2.71. And so on. The nice thing about equation (13) is that it agrees with the simulation and that it gives us a closed form. This helps us to believe the z-transform method.

As we noted, the DC gain is $1/(1-a)$ which is 10 in this example. Nothing much prevents us from putting a gain of $(1-a) = 1/10$ in equation (11) so that the DC gain becomes 1 (Fig. 4 goes asymptotically to 1 instead of to 10). Our discomfort with this is probably just that we did not have to do this in the continuous-time case.



REFERENCES:

[1] B. Hutchins, "Looking Again at the RC Low-Pass," ELECTRONOTES, Volume 22, Number 210 May 2012

<http://electronotes.netfirms.com/EN210.pdf>

[2] B. Hutchins, "Step Response from Laplace Transform," Electronotes Application Note No. 414, Aug 15, 2014

<http://electronotes.netfirms.com/AN414.pdf>

[3] This is described in many places. The "trick" of summing the infinite geometric series (or finite geometric series) is to multiply the series by $(1 - Q)$ where Q is something like z or z^{-1} or az or az^{-1} such that all the terms except, typically one (infinite series) or two (finite series), cancel. There is usually more than one choice. To give just one example, see "FIR Filters with Exponential Taps", Electronotes Application Note No. 262, Oct. 20, 1982.