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STEP RESPONSE FROM LAPLACE TRANSFORM

In the last app note, AN-413 [1] we looked at notch filters and paid special attention to the step response. We found a good number of possible ways of determining the step response, all of which happily agreed. Yet we did not use the most classic way of finding a step response of a filter T(s): by finding the Inverse Laplace Transform (LT) of T(s)/s. Like most LT problems, the procedure can be tedious.

Here we will look at the step response of a low-pass filter (one of the component responses of a notch). This is in fact a problem we did some 36 years ago in a previous app note, AN-103 [2].   Happily the result agrees with the most recent. At the same time, we get to compare the way our new tools make the job easier than it was way back then. One of the unexpected things was that for the original case, it was hard to get an answer written down that was easy to plot and verify. Today we can program "un-simplified" intermediate expressions and let a program immediately grind out and display the result. An additional goal of the present note is to make sure we understand the "calibration" of the time axis with relation to the step response.

The general form of a 2nd-order low-pass transfer function is:

$$T(s) = \frac{\omega_0^2}{s^2 + Ds\omega_0 + \omega_0^2} \tag{1}$$

where D is the damping (1/Q) while ω_0 is the pole radius. The damping can be any value greater than 0, but typically is between 0 and 2 (complex poles).   For lower values of D (say 1/2 or smaller) the step response (or the impulse response) will "ring", looking quite a bit like a superimposed sinusoidal waveform. The frequency of the sinusoidal ringing is ω_0 in radians/second. In many cases, ω_0 is "normalized" to 1 radian/second.   Thus it takes 2π seconds to complete one cycle. So if the step response for this normalized case is s(t), it takes 6.28… seconds for one full cycle.   Another significant aspect of ω_0 is that the frequency response, usually taken to be the magnitude of T(s=jω), will have a bump or peaking in the vicinity of ω_0 if D is less than $\sqrt{2}$ (like 1 or less). We look at all these things we expect them to check with each other.

The first step in doing an inverse LT problem is to find the poles so that we can write T(s) in terms of partial fractions. Setting the denominator equal to zero and using the quadratic formula, we get:

$$p_1, p_2 = \frac{-D\omega_0}{2} \pm \frac{j\omega_0}{2}\sqrt{4 - D^2} \tag{2}$$

which may look a bit odd as we have taken a "j" outside the radical sign and reversed the signs of the terms inside. We can always do this. Here it is convenient because we are thinking of cases where D is no larger than 2, and we are most interested in the corresponding complex conjugate poles. Note that the poles are stable for positive D (negative half-plane poles). For the normalized case, set $\omega_0 = 1$. Accordingly, we see that a particular choice of ω_0 simply scales the poles in or out, at an angle determined by D.

PARTIAL FRACTIONS

Our goal is to invert the LT of the step response:

$$\frac{1}{s}T(s) = \frac{\omega_0^2}{s(s^2 + Ds\omega_0 + \omega_0^2)} \qquad (3a)$$

For the normalized case $\omega_0 = 1$, this becomes:

$$\frac{1}{s}T(s) = \frac{1}{s(s^2 + Ds + 1)} \qquad (3b)$$

This inversion will be done by putting equation (3b) in the form of partial fractions:

$$\frac{1}{s}T(s) = \frac{1}{s(s-p_1)(s-p_2)} = \frac{A}{s} + \frac{B}{s-p_1} + \frac{C}{s-p_2} \qquad (4)$$

where the normalized poles are now at:

$$p_1, p_2 = \frac{-D}{2} \pm \frac{j}{2}\sqrt{4 - D^2} \qquad (5)$$

note that $p_1p_2=1$ and A, B, and C are coefficients to be determined. So the next step is to continue to develop equation (4) as:

$$\frac{A}{s} + \frac{B}{s-p_1} + \frac{C}{s-p_2} = \frac{s^2(A+B+C) - s(Ap_2 + Ap_1 + Bp_2 + Cp_1) + Ap_1p_2}{s(s-p_1)(s-p_2)} \qquad (6)$$

Equating powers of s [middle term of equation (4) to the right-side term of equation (6)] we find three equations in three unknowns and immediately see that $Ap_1p_2 = 1$ or A=1 (power of s^0), leaving two equations in two unknowns:

$$1 + B + C = 0 \qquad (power\ of\ s^2) \qquad (7a)$$
$$p_2 + p_1 + Bp_2 + Cp_1 = 0 \qquad (power\ of\ s^1) \qquad (7b)$$

Which solve for B and C to give us in total:

$$A = 1 \qquad (8a)$$
$$B = \frac{p_2}{p_1-p_2} \qquad (8b)$$
$$C = -\frac{p_1}{p_1-p_2} \qquad (8c)$$

AN-414 (2)

So we are close to having a computable form.   We have only to invert the three terms in the sum (each now first-order) individually. Likely you remember that these single-order poles invert to exponentials. But is it e^{pt} or e^{-pt} where p is the pole! Probably e^{-pt} looks right (decaying), but p probably already has a negative real part (stable). Three things to do: First, if you try one, and it blows up, you guessed wrong! A second thing is to remember that a first-order low-pass is T(s) = (1/RC)/(s + 1/RC), so the pole is at -1/RC. (See Appendix). At the same time, the exponential decay of this filter is $e^{-t/RC}$, so e^{pt} must be right. Third, all else failing, tables always have 1/(s+a)$\longleftrightarrow$$e^{-at}$ or 1/(s-a)$\longleftrightarrow$$e^{at}$. So we plug in A, B, and C:

$$\frac{1}{s}T(s) \;=\; \frac{1}{s(s-p_1)(s-p_2)} = \frac{1}{s} \;+\; \frac{\frac{p_2}{p_1-p_2}}{s-p_1} \;-\; \frac{\frac{p_1}{p_1-p_2}}{s-p_2} \tag{9a}$$

which can be inverted, term by term at first-order (the LT being linear). Calling s(t) the step response, the inverse LT of T(s)/s is:

$$s(t) \;=\; 1 \;+\; \frac{p_2}{p_1-p_2}\,e^{p_1 t} \;-\; \frac{p_1}{p_1-p_2}\,e^{p_2 t} \tag{9b}$$

This is the answer. [Note that here 1/s = 1/(s-0) inverts to 1•e^{-0t} = 1.]   We can compute it, for t = 0 to 20, for example. Here is some Matlab code:

```
% AN414.m

D=1/2
poles=roots([1 D 1]);
p1=poles(1)
p2=poles(2)
t=0:.01:20;
sstep = step([0 0 1],[1 D 1],t);    % Matlab step function for comparison
scomp = 1 +(p2/(p1-p2))*exp(p1*t) - (p1/(p1-p2))*exp(p2*t);   % eqn (9b)
figure(1)
plot(t,sstep,'r')
hold on
plot(t,scomp,'b:')
plot([-2 22],[0 0],'k')
plot([0 0],[-1 5],'k')
plot([-2 0],[0 0],'b')
plot([0 0],[0,1],'b')
plot([0 22],[1 1],'b')
plot([2*pi 2*pi],[-1 5],'c:')
plot([4*pi 4*pi],[-1 5],'c:')
tt=[2*pi:2*pi/100:4*pi];
yc=0.25 - 0.1*cos(tt);      % comparison cosine segment
plot(tt,yc,'b')
ymax=max(sstep);
axis([-2 22 -0.2 ymax+0.2])
hold off
figure(1)
```

Fig. 1

D=1/2
t=0:0.01:20
sstep = step([0 0 1],[1 D 1],t)
scomp = 1 + (p2/(p1-p2))*exp(p1*t)
            - (p1/(p1-p2))*exp(p2*t)

Comparison of Matlab
step to equation (9b)

piece of $-\cos(\omega_0 t)$
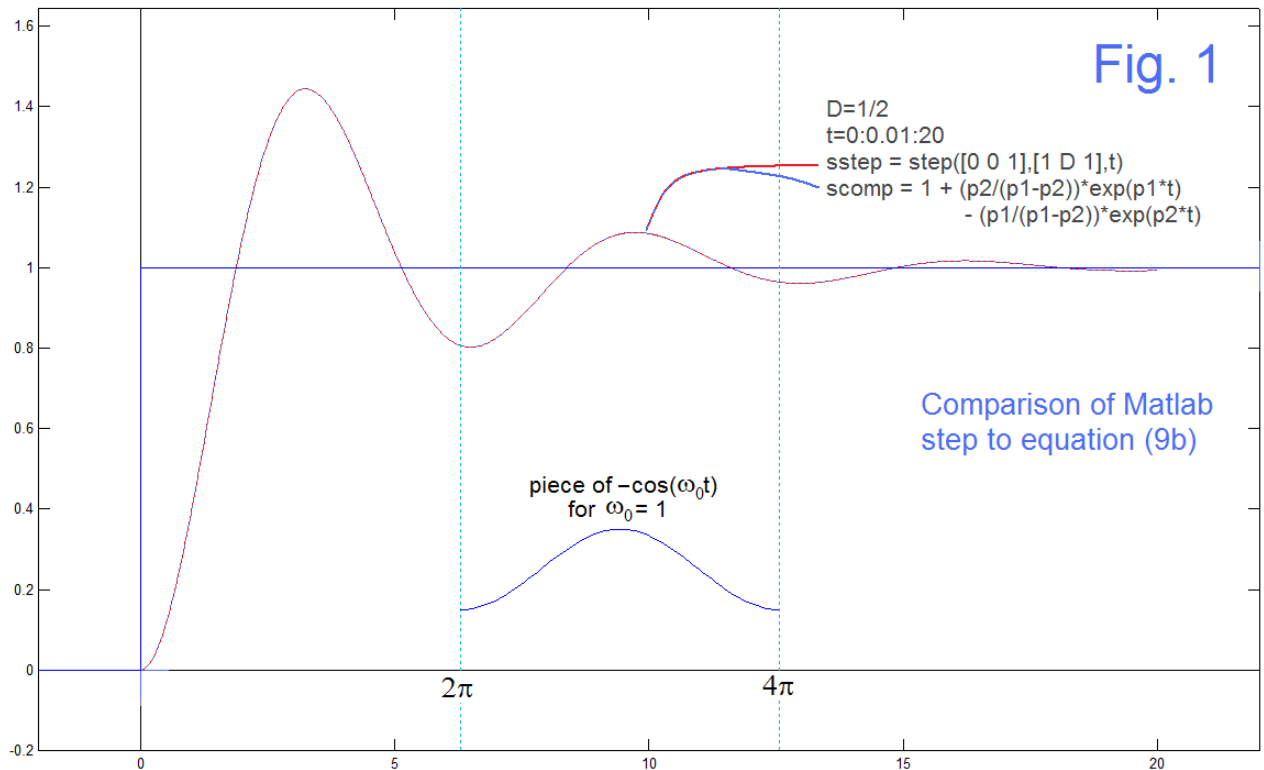for $\omega_0 = 1$

2π

4π

Fig. 1 shows the plot from this code. The first thing to note is that the curve from equation (9b) rather exactly overplots the curve from Matlab's ***step*** function. We have in the plot a superposition of a red curve with a dashed blue curve. This is the same step response we saw [1] and it agreed with other calculations. So here we have added the use of the partial fractions inverse LT procedure to our previous toolkit. We have added the little inset section to remind ourselves that the step response "rings" or oscillates resembling a frequency of $\omega_0 = 1$, something we said was useful in "calibrating" our time axis.

GRINDING IT OUT TO BE REAL

Despite the fact that the calculation of equation (9b) seems to work perfectly, it does not exactly satisfy us because it involves coefficients and exponentials that are complex, while the result must be, and is, purely real.   Some imaginary stuff must be cancelling out or somehow disappearing. While this sort of result is not at all uncommon, it is always comforting to see the details of <u>how</u> it happens.   That is, we would like to further develop equation (9b) until it is purely real. Yes, it may be tedious.

So to progress with equation (9b) we need to actually plug in the expressions for the poles from equation (5).   The expressions for the coefficients will become:

$$\frac{p_2}{p_1 - p_2} = \frac{Dj}{2\sqrt{4 - D^2}} - \frac{1}{2} \qquad (10a)$$

$$\frac{p_1}{p_1 - p_2} = \frac{Dj}{2\sqrt{4 - D^2}} + \frac{1}{2} \qquad\qquad (10b)$$

(one is the negative of the conjugate of the other) and the exponentials are:

$$e^{p_1 t} = e^{-\left(\frac{D}{2}\right)t} e^{\left(\frac{j}{2}\right)\sqrt{4-D^2}t} \qquad\qquad (11a)$$

and

$$e^{p_2 t} = e^{-\left(\frac{D}{2}\right)t} e^{-\left(\frac{j}{2}\right)\sqrt{4-D^2}t} \qquad\qquad (11b)$$

so the elements in the equation remain complex when written out, but we do see the promising symmetry which may lead to simplifying combinations. So we have:

$$s(t) = 1 + \left[\frac{Dj}{2\sqrt{4-D^2}} - \frac{1}{2}\right]\left[e^{-\left(\frac{D}{2}\right)t} e^{\left(\frac{j}{2}\right)\sqrt{4-D^2}t}\right] - \left[\frac{Dj}{2\sqrt{4-D^2}} + \frac{1}{2}\right]\left[e^{-\left(\frac{D}{2}\right)t} e^{-\left(\frac{j}{2}\right)\sqrt{4-D^2}t}\right] \qquad (12)$$

$$= 1 + \frac{Dj}{2\sqrt{4-D^2}} e^{-\left(\frac{D}{2}\right)t}[e^{\left(\frac{j}{2}\right)\sqrt{4-D^2}t} - e^{-\left(\frac{j}{2}\right)\sqrt{4-D^2}t}] - \left(\frac{1}{2}\right)e^{-\left(\frac{D}{2}\right)t}[e^{\left(\frac{j}{2}\right)\sqrt{4-D^2}t} + e^{-\left(\frac{j}{2}\right)\sqrt{4-D^2}t}] \qquad (13)$$

$$= 1 - \frac{D}{\sqrt{4-D^2}} e^{-\left(\frac{D}{2}\right)t} \sin\left(\sqrt{1 - \frac{D^2}{4}}\, t\right) - e^{-\left(\frac{D}{2}\right)t} \cos\left(\sqrt{1 - \frac{D^2}{4}}\, t\right) \qquad (14)$$

where we have used the familiar Euler relationships:

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \qquad and \qquad \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad (15a, b)$$

These Euler equations are what gets the job done here.   Equation (14) is clearly purely real and we see that the step response is the sum of a constant (the step = 1) and two exponentially <u>decaying sinusoidal</u> terms – something that we probably do not see in equation (9b). The result of equation (14) is easily programmed as:

```
scomp2 = 1 -(D/sqrt(4-D^2))*exp(-D*t/2).*sin(sqrt(1-D^2/4)*t)
                - exp(-D*t/2).*cos(sqrt(1-D^2/4)*t);
```

and gives us the same result as seen in Fig. 1.

JUDGING THE TIME RESPONSE

We want to do a few more things with Laplace, since that is all that is new here relative to AN-413, but first, let's be sure we understand the "calibration" of the time response. Fig. 2 (all using Matlab's **step**) show us that, for $\omega_0 = 1$, D=1/2, the step responses have "first bumps" that are roughly at a time of π/2 for Band-Pass (BP) and Notch and at roughly π for Low-Pass (LP) and for High-Pass (HP).   The "time" here can be considered actual time, or phase angle for $\omega_0 = 1$.

Fig. 2
Step Responses
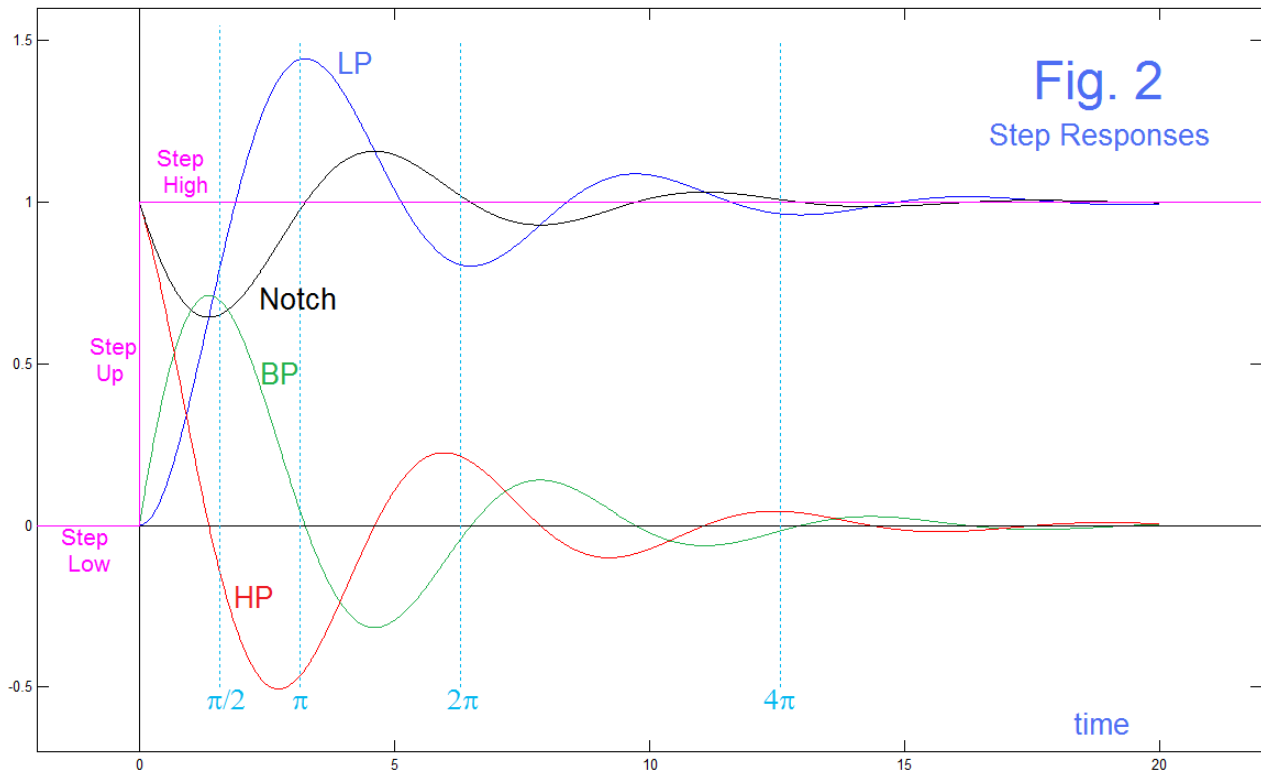
STEP RESPONSE OF HIGH-PASS BY LAPLACE

We found that calculating the step-response of a LP by the Laplace method was feasible but ever so slightly tedious. Mainly we wanted to see if it agreed with other methods. It did. We can remark here briefly on the use of the Laplace transform for HP. Note the curiosity that:

$$\left(\frac{1}{s}\right) T_H(s) = \frac{s^2}{s(s^2 + Ds + 1)} = \frac{s}{(s^2 + Ds + 1)} = T_B(s) \tag{16}$$

or the step-response of the HP will be the same as the impulse response of a BP. Thus:

$$\frac{s}{(s^2 + Ds + 1)} = \frac{s}{(s - p_1)(s - p_2)} = \frac{A}{s - p_1} + \frac{B}{s - p_2} = \frac{As - Ap_2 + Bs - Bp_1}{(s - p_1)(s - p_2)} \tag{17}$$

From which (equating powers of s in the numerator) we have:

$$A + B = 1 \tag{18a}$$

$$Ap_2 + Bp_1 = 0 \tag{18b}$$

which solve easily for:

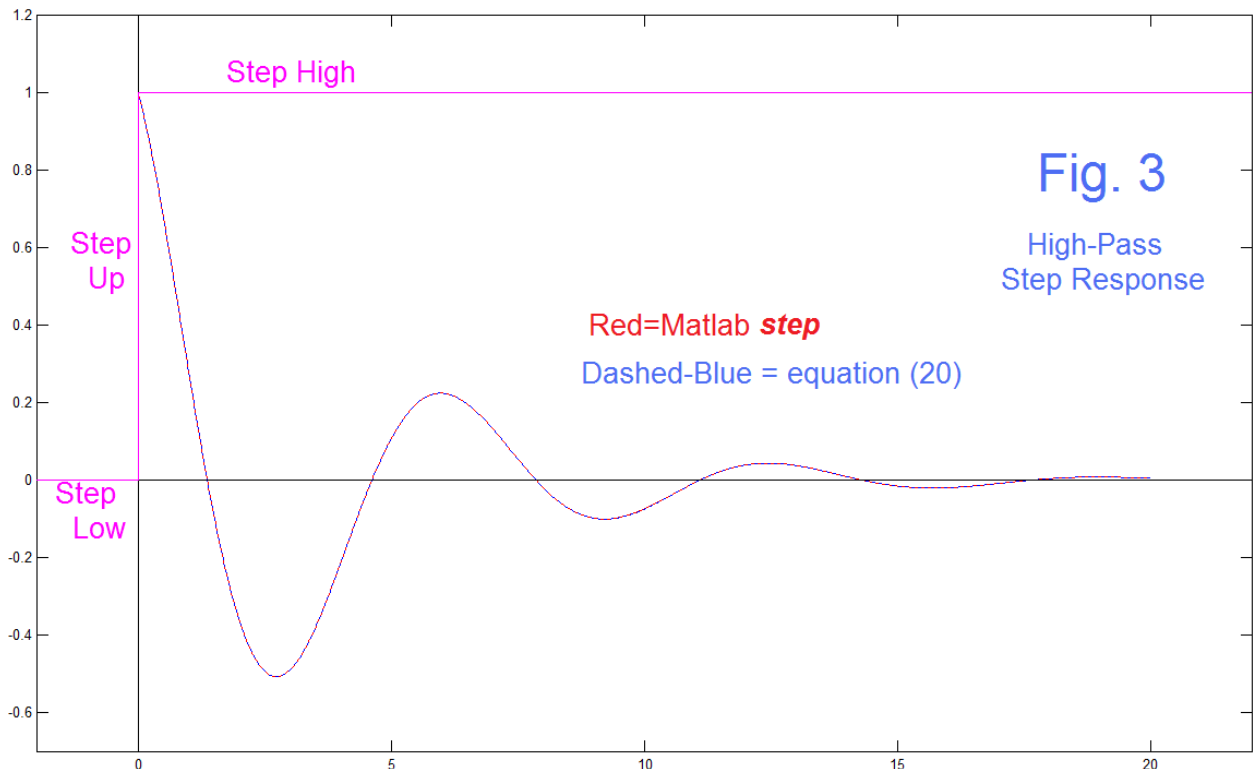$$A = \frac{-p_1}{p_2 - p_1} \quad and \quad B = \frac{p_2}{p_2 - p_1} \tag{19a,b}$$

so $s_{HP}(t)$, the high-pass step response is:

$$s_{HP}(t) = \frac{-p_1}{p_2 - p_1} e^{p_1 t} + \frac{p_2}{p_2 - p_1} e^{p_2 t} \tag{20}$$

which is programmed as:

```
scomphp=(-p1/(p2-p1))*exp(p1*t) + (p2/(p2-p1))*exp(p2*t);
```

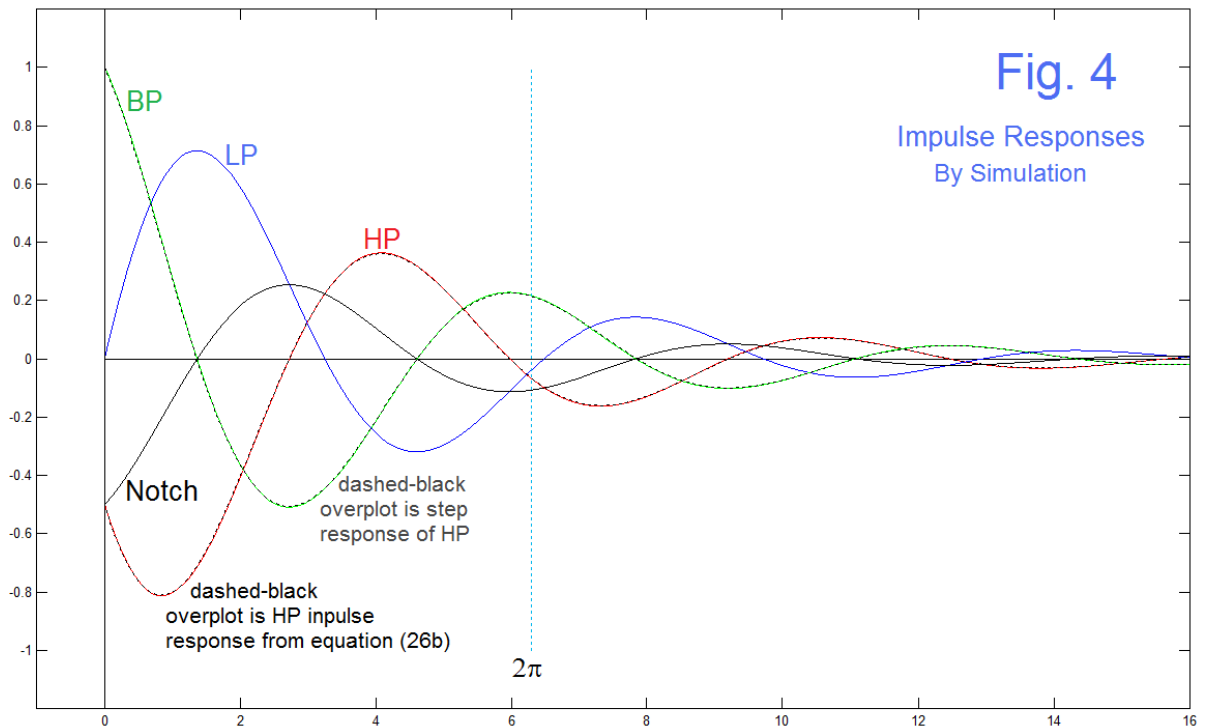and can be plotted as in Fig. 3 (D=1/2), along with the Matlab *step* result.

IMPULSE RESPONSE OF HIGH-PASS BY LAPLACE

While we have concentrated on step-responses, the LT is probably most used for impulse responses: the impulse response g(t) being the inverse LT of T(s). We want to show here the complications of applying the methods to the HP case, but first we will calculate the impulse responses of the LP, BP, HP, and Notch using simulation. Fig. 4 shows the results (with two dashed overplots to be described), and the Matlab code (similar to the step response code of AN-413), is shown below:

AN-414 (7)

```
VH=zeros(1,8000);
VB=zeros(1,8000);
VL=zeros(1,8000);
dt=0.002
D=1/2
imp=500
for n=2:8000
    VL(n)=VL(n-1)+VB(n-1)*dt;
    VB(n)=VB(n-1)+VH(n-1)*dt;
    VH(n)=-VL(n-1)-D*VB(n-1)+ imp;
    imp=0;
end
VN=VH+VL;
figure(6)
plot([0,20],[0 0],'k')
hold on
plot([0 0],[-2 2],'k')
plot([0:7998]/500,VH(2:8000),'r')
plot([0:7998]/500,VL(2:8000),'b')
plot([0:7998]/500,VN(2:8000),'k')
plot([0:7998]/500,VB(2:8000),'g')
plot(t,scomphp,'k:')
plot(t,hpir,'k:')
plot([2*pi 2*pi],[-1 1],'c:')
axis([-1   16  -1.2 1.2])
hold off
title('Impulse Responses LP=Blue  HP=Red  BP=Green  N=Black')
figure(6)
```


Fig. 4

Impulse Responses
By Simulation

BP

LP

HP

Notch

dashed-black
overplot is step
response of HP

dashed-black
overplot is HP inpulse
response from equation (26b)

2π

Less we have left the impression that the LT method is without complications, we will look at the case of finding the impulse response of the HP using the partial fractions approach. Something different happens here. Suppose we try:

$$T_H(s) = \frac{s^2}{(s^2 + Ds + 1)} = \frac{s^2}{(s-p_1)(s-p_2)} = \frac{A}{s-p_1} + \frac{B}{s-p_2} = \frac{As - Ap_2 + Bs - Bp_1}{(s-p_1)(s-p_2)} \qquad (?) \qquad (21)$$

This can't possibly work, because the numerator on the far right of equation (21) has no terms in s^2. We must add a constant term, and this in turn means that the inverse transform will have a peaky Dirac delta function. Thus we try instead:

$$T_H(s) = \frac{s^2}{(s^2 + Ds + 1)} = \frac{s^2}{(s-p_1)(s-p_2)} = A + \frac{B}{s-p_1} + \frac{C}{s-p_2} = \frac{A(s^2 + Ds + 1) + B(s-p_2) + C(s-p_1)}{(s-p_1)(s-p_2)} \qquad (22)$$

So the numerator on the far right term of equation (22) is:

$$s^2 A + s(AD + B + C) + A - Bp_2 - Cp_1 \qquad (23)$$

from which we match to the s^2 term to give A=1, with two equations in two unknowns remaining:

$$AD + B + C = D + B + C = 0 \qquad (24a)$$

$$A - Bp_2 - Cp_1 = 1 - Bp_2 - Cp_1 = 0 \qquad (24b)$$

So our coefficients become:

$$A = 1 \qquad (25a)$$

$$B = \frac{-(1 + Dp_1)}{p_1 - p_2} \qquad (25b)$$

$$C = \frac{(1 + Dp_2)}{p_1 - p_2} \qquad (25c)$$

So the partial fraction expansion of $T_H(s)$ is:

$$T_H(s) = 1 + \frac{[\frac{-(1 + Dp_1)}{p_1 - p_2}]}{s - p_1} + \frac{[\frac{(1 + Dp_2)}{p_1 - p_2}]}{s - p_2} \qquad (26a)$$

with corresponding impulse response:

$$g_H(t) = \delta(t) + \frac{-(1 + Dp_1)}{p_1 - p_2} e^{p_1 t} + \frac{(1 + Dp_2)}{p_1 - p_2} e^{p_2 t} \qquad (26b)$$

AN-414 (9)

Equation (26b) is overplotted in Fig. 4 as a dashed black line overploting the red simulation curve, with an excellent matching. Here is the Matlab code line for the HP impulse response of equation (26b)(The Dirac delta can not be plotted):

```
hpir=((-1-D*p1)/(p1-p2))*exp(p1*t) + ((1+D*p2)/(p1-p2))*exp(p2*t);
```

Fig. 5 shows an experimentally observed curve for the same circuit as in AN-413 Fig. 7 (except D=1/2, so the 68k resistor there was changed to 200k here), and the input was a narrow (but not too narrow so that some energy gets in, pulse approximating an impulse). Not the easiest of experiments.
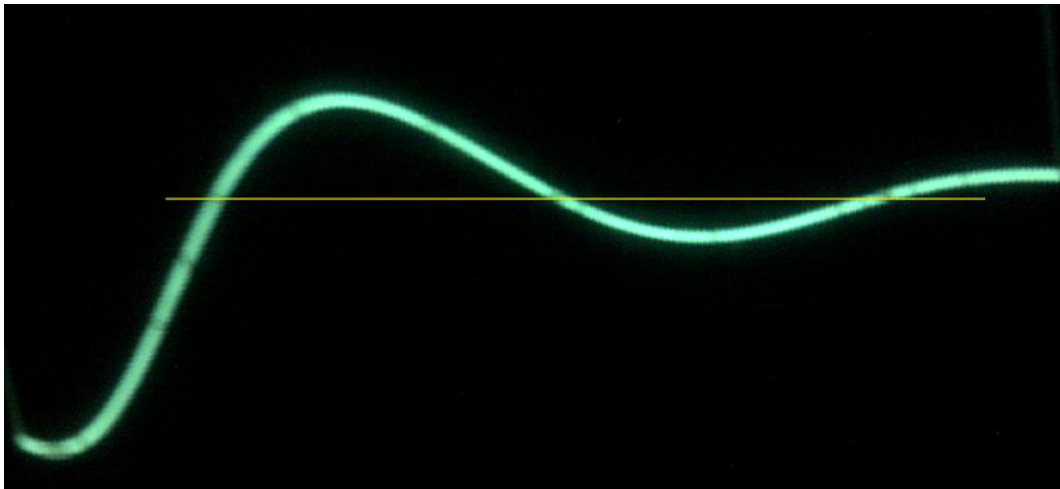

Fig. 5
High-Pass
Impluse
Response
D=1/2

The agreement here is quite good.

REFERENCES:

[1] "Notch Filtering – Some Notes", Electronotes Application Note No. 413, July 30, 2014
http://electronotes.netfirms.com/AN413.pdf

[2] "Second-Order Step Response", Electronotes Application Note No. 103, Sept 18. 1978
http://electronotes.netfirms.com/AN103.PDF
This note used tables (CRC Math Tables) for the LT pair. The expression in AN-103, equation (7), computes just fine to give the same results as here. There is possibly an overall (-) sign in the tables which is curious. Worth tracking down some day – perhaps.

APPENDIX – FIRST-ORDER

As mentioned, there is at time confusion about the signs of poles (sign in the denominators) and in the exponentials. In addition to generally noticing if we did it wrong, because the result blows up, we suggest that one can resort to a first-order LP, which is likely very familiar. We know for example that the pole is real and negative (at -1/RC, stable), and that the impulse response should decay exponentially to zero. The figure below reminds us that we can think of an impulse as the instantaneous transfer of an amount of charge q_0 to a capacitor at time 0. with $q_0C=1$. In terms of a Dirac delta, this is a very large ("infinite") pulse of zero width, but unit area! For all time except exactly t=0, the "input" is in effect, grounded, so the capacitor discharges exponentially for positive time. It is exactly the same discharge we would get if the input had been 1 from t = -∞ to t=0, and then suddenly fell to zero.
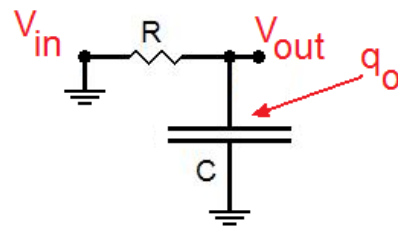
voltage-divider

$$T(s) = \frac{1/sC}{R + 1/sC} = \frac{1/RC}{s + 1/RC}$$

pole at s = p = -1/RC

"p" already has negative real part

$$\frac{V_{out}}{Cq_o} = e^{-t/RC}$$

1/e
0.368

time in units of RC