**ELECTRONOTES**
1016 Hanshaw Road
Ithaca, NY  14850

# APPLICATION NOTE NO. 410

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## FOURIER MAP

**For Reference:** The first map (this page) was published in EN#188 (1993). The second map (with comments, pages (2)-(4) here, was produced in 2006 and used for teaching purposes.

### TABLE 1  FOURIER MAP

Six major transforms can be related in terms of whether or not their domains are continuous (c) or discrete (d) and whether they are periodic (p) or non-periodic (n). Periodicity in one domain forces the other domain to be discrete.

time/frequency duals

**Fourier Series**

$$c(k) = (1/P)\int_{-P/2}^{P/2} f(t)\, e^{-2\pi jkt/P}\, dt$$

(d,n)

$$f(t) = \sum_{k=-\infty}^{\infty} c(k)\, e^{2\pi jkt/P}$$

(c,p)

← Signal Analysis | System Analysis →

**Fourier Transform**

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

(c,n)

$$f(t) = (1/2\pi)\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

(c,n)

← jω-axis only

**Laplace Transform**

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

$$f(t) = (1/2\pi j)\oint F(s) e^{st}\, ds$$

Continuous Time ↓ Discrete Time

**Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jn\omega T}$$

(c,p)

$$x(n) = (1/\omega_s)\int_{-\omega_s/2}^{\omega_s/2} X(\omega)\, e^{jn\omega T}\, d\omega$$

(d,n)

← unit circle only

**z-Transform**

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$x(n) = (1/2\pi j)\oint X(z) z^{n-1}\, dz$$

↓ Sample in Frequency ↓

**Discrete Fourier Transform (DFT)**

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j(2\pi/N)nk}$$

(d,p)

$$x(n) = (1/N)\sum_{k=0}^{N-1} X(k)\, e^{j(2\pi/N)nk}$$

(d,p)

Notes:
- FFT = fast DFT
- P = period of f(t)
- $z = e^{sT}$
- $T = 1/f_s$
- $f_s$ = sampling frequency
- $\omega_s = 2\pi f_s = 2\pi/T$

From EN#188 (1993)

# FOURIER MAP

## CTFT

Continuous-Time Fourier Transform

$$F(\Omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\Omega t}\, dt$$

[continuous, non-periodic]

$$f(t) = (1/2\pi) \int_{-\infty}^{\infty} F(\Omega) e^{j\Omega t}\, d\Omega$$

[continuous, non-periodic]

## FOURIER SERIES

$$c(k) = (1/P) \int_{-P/2}^{P/2} f(t)\, e^{-j2\pi kt/P}\, dt$$

[discrete, non-periodic]

$$f(t) = \sum_{k=-\infty}^{\infty} c(k)\, e^{j2\pi kt/P}$$

[continuous, periodic]

## DTFT

Discrete-Time Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\omega}$$

[continuous, periodic]

$$x(n) = (1/2\pi) \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{jn\omega}\, d\omega$$

[discrete, non-periodic]

## DFT

Discrete Fourier Transform
(fast version is Fast Fourier Transform FFT)

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j(2\pi/N)nk}$$

[discrete, periodic]

$$x(n) = (1/N) \sum_{k=0}^{N-1} X(k)\, e^{j(2\pi/N)nk}$$

[discrete, periodic]

**Notes on Fourier Map**

(1) There are four Fourier-transform-like pairs here that describe the same functions in the time domain (t or n) and in the frequency domain (Ω, ω, or k)

(2) One of the pairs (CTFT) is an integral/integral pair, one (DFT) is a summation/summation pair, and the remaining two involve an integral/summation pairing.

(3) Both time and frequency descriptions may be either continuous or discrete (hence the four possible cases. To see if the description is continuous or discrete, consider first whether the equation is an integral (continuous) or a summation (discrete). Also, the frequency variable for continuous time is t, while that for discrete time is n. For frequency, discrete frequency is k, while continuous frequency is Ω or ω.

(4) The continuous frequency variable Ω is considered to be in physical units of radians/sec. Thus frequency in Hertz would be $\Omega/2\pi$. The continuous frequency ω has various interpretations. One is that it is radians/second where the sampling frequency is set to 1 Hz ($2\pi$ radian/second, a normalized sampling time T=1). In this view, the sampling frequency in Hertz is $f_s$, the sampling time is $T=1/f_s$, and $\omega = \Omega T$. We could rewrite the equations for the DTFT using ΩT for ω. Possibly the most useful interpretation of ω is that it is a dimensionless angle. It is the angle in the unit circle in the z-plane. Sometimes this is called radians/sample.

(5) Note that for each pair, when the description in one domain is periodic, the description in the other domain is discrete (think Fourier Series). When the description in one domain is non-periodic, the description on the other domain is continuous.

(6) The Fourier Series and the DTFT are "cross" cases where [discrete, non-periodic] is paired with to [continuous, periodic]. Mathematically, they are the same except the roles of time and frequency are reversed (plus a few minor notational conventions).

(7) The DTFT is thus a Fourier Series for a periodic function of frequency and the "Fourier Series coefficients" are the discrete samples in time. Thus the DTFT is always periodic (with a period, the spacing of the sampling replicas) equal to the sampling frequency (which is $2\pi$ for the normalized case on the map). This periodicity in frequency is the direct consequence of sampling in time.

(8) For the DTFT, we might think of x(n) as a signal or it might be the "impulse response" of a filter, which we would probably denote h(n). The DTFT of a signal is usually called its "Spectrum" while the DTFT of an impulse response is called the "Frequency Response." Mathematically there is no difference, but use the terminology right: a <u>s</u>ignal has a <u>s</u>pectrum while a <u>f</u>ilter has a <u>f</u>requency response. If at some point you become confused about the nature of the DTFT, it is often useful to think of it in terms of the nature of a frequency response (a prime example of a DTFT).

(9) The top two boxes relate to continuous time, while the bottom two relate to discrete time. Thus time sampling occurs in the bottom boxes. The boxes on the left relate to continuous frequency (like turning the knob on a function generator) while the boxes on the right relate to sampling in frequency. The DFT is a sampling (samples at frequencies $2\pi/N$) of the DTFT.

(10) As noted the bottom/left to top/right pairs are "Fourier Series" cases. Looking now instead at the top/left and bottom/right pairs we see that the DFT may be thought of as a sort of numerical integration of the CTFT. However, this numerical integration means that both time and frequency have to be sampled, and hence are periodic, so this complicates our thinking.

(11) Where is the famous "Laplace transform?" The LT is the father of all the transforms here. The LT relates to the entire "s-Plane" while the Fourier transforms here relate only to the imaginary axis in the s-plane ("real" or physical frequencies).

(12) And where is the "z-Transform?" This makes the LT the grandfather of the DTFT and the DFT. The z-transform is the LT evaluated for a sampled time function, and the z-transform applies to the entire "z-plane." The DTFT applies only to the unit circle in the z-plane and the DFT is further restricted to only discrete points on that unit circle.

(13) And what about the very famous Fast Fourier Transform (FFT)? The FFT is a fast algorithm for computing the DFT. The FFT (as an algorithm) is not the DFT (a brute-force summation). But the <u>FFT of</u> a sequence x(n) is identical to the <u>DFT of</u> the same sequence x(n). It is not an approximation. Because the two results are the same, the terms are often used interchangeably. It perhaps makes sense to use FFT rather then DFT because (a) of the danger of confusing the terminology DFT with DTFT, and (b) because virtually all DFT's are actually computed by FFT. In fact, virtually all the transforms in the four boxes are (or can be, or for practical purposes must be) computed using the FFT.