ELECTRONOTES
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APPLICATION NOTE NO. 400

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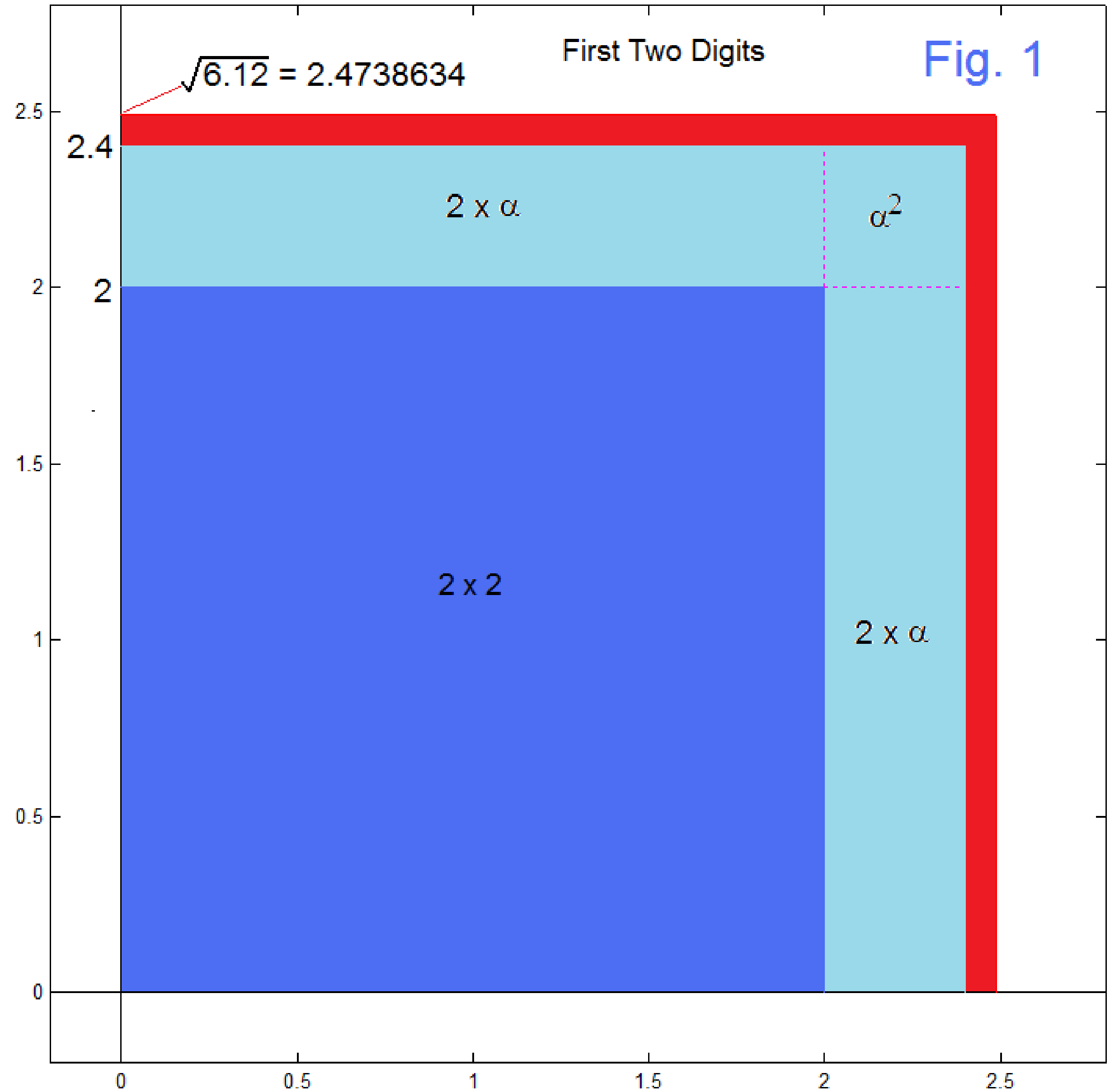
SQUARE-ROOTS BY HAND

Back in the "good old" days we used to calculate square-roots by hand. In my case, I guess this was the late 1950's. It was clear enough what a square root was. It was also clear that one could estimate a square root by trial and error. Close by was the notion that some square-roots went on forever (were irrational), and far in our future education were square-roots of negative numbers ("imaginary" numbers). A treasure trove awaited.

What we had were no computers or calculators, and only sparse tables of some square-roots. And there was a formal method of calculating a square root digit by digit. It involved the radical sign (√) and a scheme that looked something like "long division". I think they did that on purpose to avoid scaring us off. Today we should see it as an iterative algorithm. To my surprise today, if we look up the "hand method" on the web, the exact method I learned does not come eagerly home. But the details vanished long ago anyway.

I went to a small school and the same teacher did most of the math along with all the sciences. Accordingly, I observed that as dogmatic as he was about "proof" in geometry, he also eschewed proof (and often even derivation) elsewhere. We did do quite well learning from example. Such was our experience when he said he didn't know how to derive the square-root method - but here is how it is done. [Even as he often avoided proofs, he and I spared almost daily, and I once gave him a proof that $\sqrt{2}$ was a rational number, and he took it home. He and I shared being "right" about 50:50, so the next day we both laughed. You were one of the best, Russell B. May, and I apologize for forgetting how to do a square root by hand.]

Today, we understand that there was an overall algorithm, the exact mechanical details of which may vary among school systems. The algorithm is one of successive approximation, starting from the low side. But it is not a method of refining a guess which would be a succession of trials, over and under. Instead we can think of it as a method of trying to reach a correct size square starting from a square that is just undersized. For example, when we want to find $\sqrt{5}$ we know it is greater than $\sqrt{4}$ = 2 but less than $\sqrt{9} = 3$. Having made this observation, we decide that the first digit of $\sqrt{5}$ is going to be 2.



To illustrate the algorithm and to show its essentially geometric nature we show Fig. 1. We have chosen to find the square-root of 6.12, which our calculator shows to be 2.4738634… , but we do not know this ahead of time. In Fig. 1 we show this as a red square of side $\sqrt{6.12}$ but we are to understand that the red "square" is overplotted with the blue and light blue regions that represent the first two digits to be calculated. The large square has area 6.12. Note that we are not talking here about "guessing" the square root and then tweaking the guess, although this would show something very similar to Fig. 1 as a visual aid. Here we are accurately calculating each digit successively. As with our case of $\sqrt{5}$, we note that 6.12 is between 2^2=4 and 3^2=9, so our first digit is 2. The darker blue square approximates the full red square. Our attention turns to the lighter blue region.

The only choices for the second digit are of course 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, and 0.9. Much as we chose 2 as the smallest integer whose square is less than 6.2, we choose 2.4 as our next estimate. At this point, it sounds a lot like guess and tweak. Here however we are only looking to choose the light blue area so as to better fill the remaining space, while not overfilling it. And the procedure is mechanical.

What is the area that is filled with the dark blue. It's $2^2=4$ of course. The full red square had an area 6.12 of course. So we have 6.12 – 4 = 2.12 to fill, potentially. What is the area of the L-shaped light blue region? Well, it depends on the width, which is our guess for the second digit, and let's call the corresponding decimal fraction α. (For example, if the second digit is a 3, α=0.3). The L-shaped light blue area has two regions of area $2\times\alpha$ and one region of α^2 or $2(2\times\alpha) + \alpha^2$. If we try different values of α. for α=0.4, the light blue area is 1.76 while of α=0.5. the light blue area is 2.25. The area we needed to make up was only 2.12, so we choose α=0.4 and our second digit is 4, so our calculation so far is 2.4. The remaining area is this 6.12 – (4 + 1.76) = 6.12 – 5.76 or 0.36. Alternatively, the area remaining is 2.12 – 1.76 or 0.36 – the same thing.

This repeats, and we now want to choose a third digit and our decimal fraction choices are 0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, and 0.09. So we envision a larger square (not shown) beyond the light blue but still in the red area. The L-shaped area now has area $2(2.4\times\alpha) + \alpha^2$. (Note that the rectangle side is the now the just-established new estimate, 2.4, and not just the 2. We still have the 2 outside the parenthesis because there are two rectangles.)

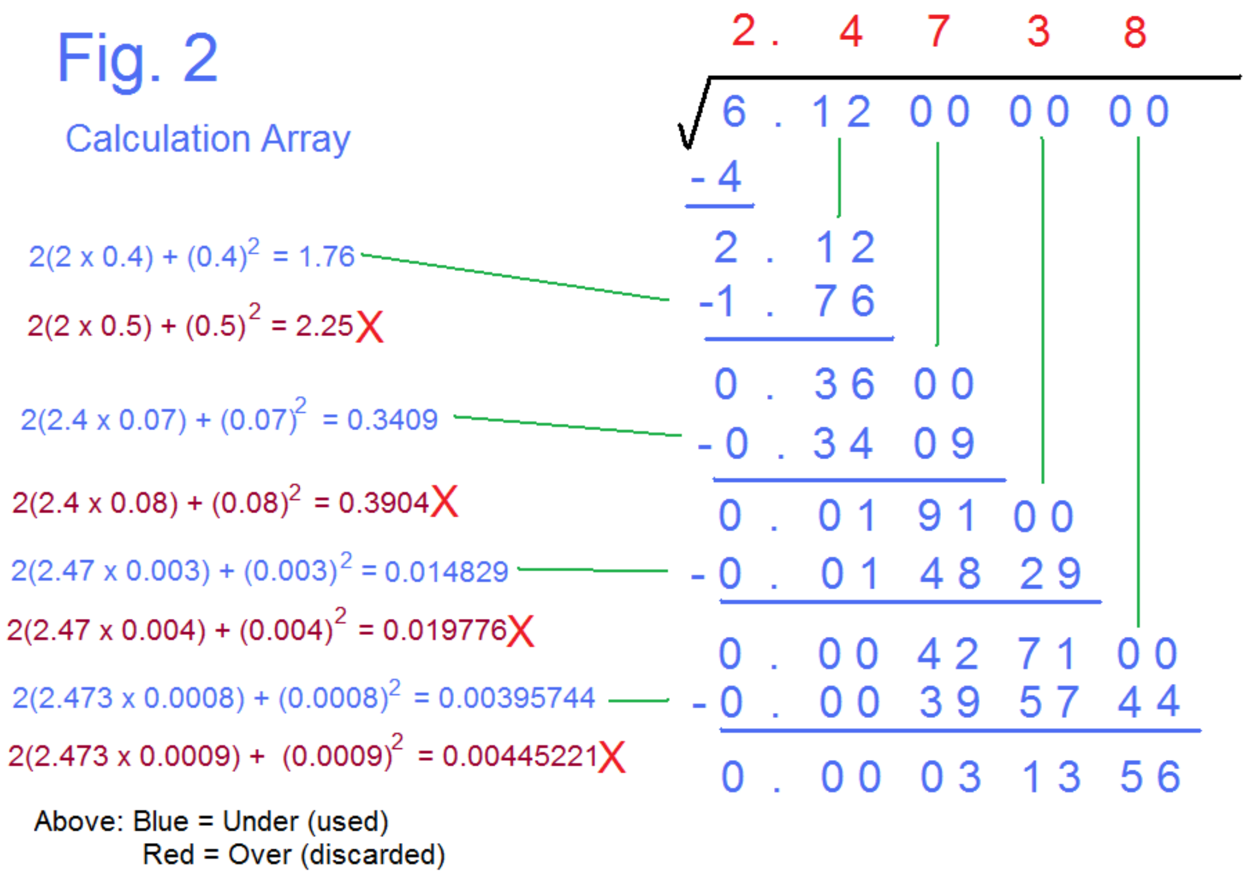
Now we have for α=0.07 an area of 0.3409 and for α=0.08 an area of 0.3904, the first being less than the 0.3600 needed and the latter being too much. Thus or third digit is a 7, and our calculation advances to 2.47. Note that $(2.47)^2$ is already 6.1009 and we are trying to get to 6.12. So things look right so far.

Here we have talked our way through the algorithm keeping in mind the geometry. I would not object too strongly to anyone who points out that there is still a "trial and error" stage here as we choose α. But this verification is to select among calculate alternative, so as to sneak up on each new digit, and not an evaluation to choose a better guess. Not that any of these case are of practical value anyway! This is history.

As I have suggested, my recollection is that the geometric picture here was not part of our math training. In consequence, what was done above is not what many of us remember. As I said, it was kind of like long division, and Fig. 2 shows one standard "array calculation" (done to 5 digits) which looks more or less familiar, depending on where you went to school. Study it carefully and see that it is really the geometric algorithm.

Fig. 2

Calculation Array



This may look a bit different from what you used – I think it is different from the "recipe" we had. Here I have kept decimal points lined up, while some methods effectively multiply by 10 at each stage going down, which avoids lots of leading zeros in the decimals, but may hide the geometric simplicity. Note that we bring down two digits at a time. This is much the same as the effect we have when 3 x 3 is 9 and 30 x 30 is 900, a two digit increase.

The information provided by Rob Edwards and Ed Kellett regarding how math was taught in the late 50's is much appreciated.