ELECTRONOTES
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APPLICATION NOTE NO. 394

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NODE VOLTAGES BY SUPERPOSITION

Once we encounter Ohm's Law, probably the first problem we are asked to solve is the voltage-divider. This shown for two impedances Z is shown in Fig. 1 below. We can solve this by calculating a current and a voltage drop across the lower leg (the one to ground) of the divider. This is:

$$V_{OUT-1} = \frac{Z_2 V_1}{Z_1 + Z_2} \tag{1}$$

Almost certainly everyone reading this would have written down equation (1) without deriving anything – it is so familiar. Further, we recognize that the actual equivalents of Z might be a resistor R, a capacitor 1/sC, or an inductor sL, OR some combination of these as long as we have only two terminals to the network. For example, Z might be a parallel combination of a Resistor R and a capacitor C, which we calculate at R / (1 + sCR). Note that we DO ASSUME that no current is flowing from the V_{OUT-1} node. Old stuff here.

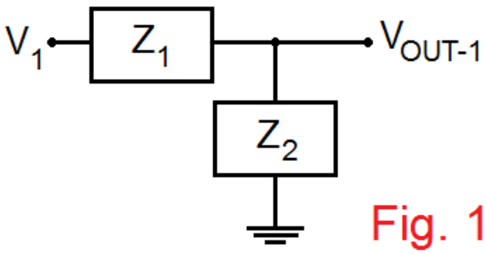


Fig. 1

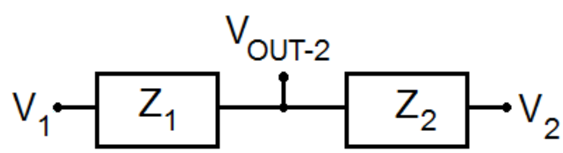


Fig. 2

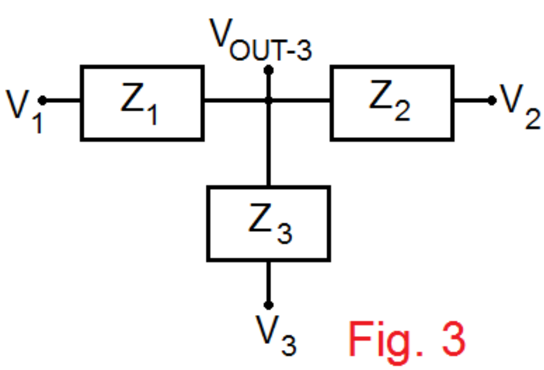


Fig. 3

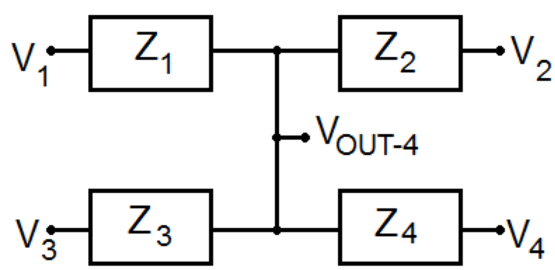


Fig. 4

Almost as familiar as the simple divider is the problem of finding the voltage at a node between two impedances such as in Fig. 2. Again we assume that no current is flowing out of the V_{OUT-2} node, although a current is generally expected to flow through it, from V_1, through Z_1, through the node, through Z_2, and to V_2. There is just no current "tapped off" of the node. We perhaps measure the voltage with a high-impedance meter or scope, or feed the voltage to an op-amp input terminal. So what is V_{OUT-2} in terms of V_1, V_2, Z_1, and Z_2?

You perhaps are tempted to calculate the current and voltage drop and attach this voltage drop to one of the input voltages. You know exactly what to do, but hopefully you will say "Hey – this is a linear system – I can just apply superposition". That is, the answer is the sum of applying V_1 with V_2 assumed to be zero and then V_2 with V_1 assumed to be zero. Applying the voltage divider in both directions. You can always work it out the hard way a few times if you doubt this. Soon enough you know that:

"The voltage in the middle is the first voltage times the impedance in the opposite leg plus the second voltage times the impedance in its opposite leg all divided by the sum of the two impedances"

$$V_{OUT-2} = \frac{V_1 Z_2 + V_2 Z_1}{Z_1 + Z_2} \tag{2}$$

Either you know this, or will soon consider it an essential and trusted shortcut. It is just using equation (1) twice. So far so good. Bag of tricks.

So what if you have three impedances connected to a node, or four, or more? Let's convince ourselves that we know how to solve the problem. (1) Superposition still applies of course. (2) And when we look at the contribution of any one voltage, with the other voltages set to zero, those set to zero are all impedances in parallel, and we can find the equivalent of the combination. Done – just the details.

Exactly how we proceed with three or more impedances probably depends on the complexity of the impedances. For example, if they are three (or more) resistors all of resistance R (any R) we know the node in the middle is just the average. This won't slow us down even one second. If on the other hand, the impedances are unequal resistors and/or parallel and/or series inductors and/or capacitors, well, this may take some time.

You might well see two possible approaches. First, you might just try to do all the algebra at once, filling a page with the various terms and hoping for the best. You best know if this looks like something you normally overcome (or likely not). You are aided by such things as "dimensional checks" (you can't add R and C, and sCR is dimensionless, etc.), but it's still tough. At the end, you can check some limiting cases (like does it give the right answer if a certain C →0). Yet dimensional errors and "limiting cases" only show that you <u>may not be wrong</u> – not that you <u>are right</u>. Why not just do the algebra a second time? Yes, observe that it is easier to do the second run – probably requiring about half the time. But the corollary to saving time is the possibility that you also followed familiar steps, and perhaps the same errors. So you are working on an "easy" problem but one which can be very tedious. Something like equation (2) or the rule written above it sure would seem handy, even if just as an alternative check.

So – moving ahead ot Fig. 3, three impedances. It is clear that because of symmetry we only have to figure out one term, so let's assume that V_1 is active and V_2 and V_3 have been set to zero. So this is really just a voltage divider with Z_1 in the top leg and the parallel combination of Z_2 and Z_3 in the lower leg. The contribution of this voltage V_1 to V_{OUT-3}, which we can call $V_{OUT-3(From\ 1)}$, using the notation [$Z_2||Z_3$] to indicate the parallel combination of Z_2 and Z_3, is:

$$V_{OUT-3(From1)} = \frac{V_1[\,Z_2||Z_3\,]}{Z_1+[\,Z_2||Z_3\,]} \tag{3}$$

But [$Z_2||Z_3$] = $(Z_2Z_3)/(Z_2 + Z_3)$ which is exactly the form we use for parallel resistors of course. Plugging this in and simplifying we have:

$$V_{OUT-3(From1)} = \frac{V_1(\,Z_2Z_3\,)}{Z_1Z_2+Z_1Z_3+Z_2Z_3} \tag{4}$$

Then summing the exactly similar contributions of V_2 and V_3, we get:

$$V_{OUT-3} = \frac{V_1(Z_2Z_3)+V_2(Z_1Z_3)+V_3(Z_1Z_2)}{Z_1Z_2+Z_1Z_3+Z_2Z_3} \tag{5}$$

and this is the answer; we just need to plug in the impedances.

It is interesting to put the results into words, as we did just above for equation (2).

"Form a numerator as the sum of three terms obtained by multiplying each voltage by the product of the impedances in the other two legs. Then divide by the sum of the products of the impedances in the numerator"

Please note at this point that this reduces to equation (2) for the limiting case where Z_3 becomes infinite. This of course should happen. And if all the impedances are the same, we get the average of the three voltages, as we should. We are almost able to guess a general rule. But we kind of need to try one more impedance to make the scheme totally obvious.

Fig. 4 shows four impedances. Now we have three parallel impedances in the far legs as we use our superposition component ploy.

$$V_{OUT-4(From1)} = \frac{V_1[\,Z_2||Z_3||Z_4\,]}{Z_1+[\,Z_2||Z_3||Z_4\,]} \tag{6}$$

We can do the three in parallel by combining two first, and then the third to that result. Perhaps it is better to just remember that impedances in parallel add as reciprocals. Calling the parallel combination $[Z_2||Z_3||Z_4] = Z^*$ we have:

$$\frac{1}{Z^*} = \frac{1}{Z_2} + \frac{1}{Z_3} + \frac{1}{Z_4} = \frac{Z_3Z_4+Z_2Z_4+Z_2Z_3}{Z_2Z_3Z_4} \tag{7}$$

so:

$$Z^* = \frac{Z_2Z_3Z_4}{Z_3Z_4+Z_2Z_4+Z_2Z_3} \tag{8}$$

and:

$$Z_1 + Z^* = \frac{Z_1Z_3Z_4+Z_1Z_2Z_4+Z_1Z_2Z_3+Z_2Z_3Z_4}{Z_3Z_4+Z_2Z_4+Z_2Z_3} \tag{9}$$

$$V_{OUT-4(From1)} = \frac{Z^*V_1}{Z_1+Z^*} = \frac{V_1(\,Z_2Z_3Z_4\,)}{Z_1Z_3Z_4+Z_1Z_2Z_4+Z_1Z_2Z_3+Z_2Z_3Z_4} \tag{10}$$

Then summing the exactly similar contributions of V_2 and V_3, we get:

$$V_{OUT-4} = \frac{V_1(Z_2Z_3Z_4) + V_2(Z_1Z_3Z_4) + V_3(Z_1Z_2Z_4) + V_4(\,Z_1Z_2Z_3)}{Z_2Z_3Z_4+Z_1Z_3Z_4+Z_1Z_2Z_4+Z_1Z_2Z_3} \tag{11}$$

This result is the extension of equations (2) and (5). Note that if we look at the case where Z_3 and Z_4 go to infinity, we keep only the terms which have the product Z_3Z_4, and we get back equation (2), as is required. Likewise, if Z_4 by itself goes to infinity, we get back equation (5). In addition, if all four impedances are identical, we get back the average of the four voltages.

The verbal rule for three impedances is thus modified for four impedances as follows:

> **"Form a numerator as the sum of four terms obtained by multiplying each voltage by the product of the impedances in the other three legs. Then divide by the sum of the products of the impedances in the numerator"**

We are in a position, now, to write a general rule for N impedances that is not that hard to remember.

> **"Form a numerator as the sum of N terms obtained by multiplying each voltage by the product of the impedances in the other N-1 legs. Then divide by the sum of the products of the impedances in the numerator"**

All you need to do is write down the sum of all the voltages multiplied by the "other" impedances and divide by the sum of the impedance products.

And then, check to be sure your result is dimensionally correct and correct in the limits.

Finally, check again.

TWO EXAMPLES

If you have reached this point, you will likely be interested in some examples, and may well have just jumped here, recognizing that the algebra above was largely a tedious derivation. We will do the examples using <u>only the rule stated in words</u> above (purple in the most general case). Fig. 5A shows the first example. In the absence of the one shunt capacitor C, this is just a

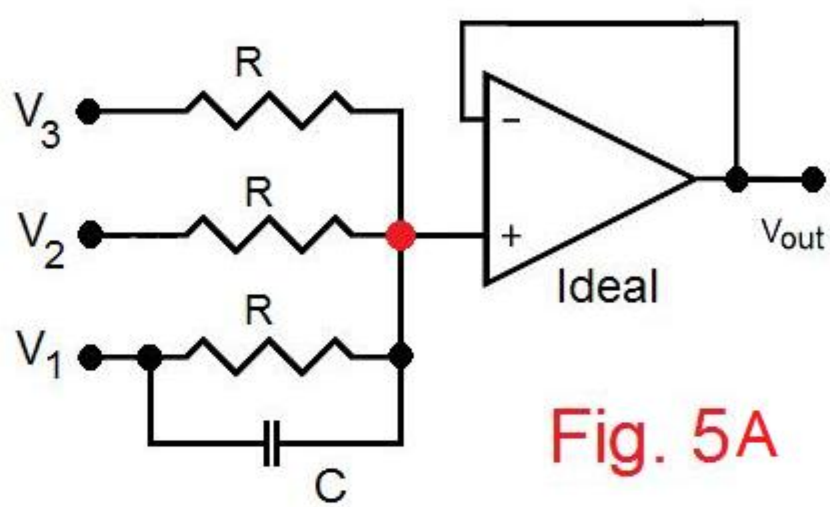


Fig. 5A

three input average. The voltage at the red node is perfectly buffered by the ideal op-amp follower, so that node voltage is already V_{out}. Perhaps we suppose that the capacitor C will just be a high-frequency boost, and apply to the V_1 node. Using the verbal rule however we can just write V_{out} as a fraction. For the numerator, we write down a summation of the three voltages V_1, V_2, and V_3, and apply a weighting to each voltage that is the product of the two impedances in the legs away from that voltage. For the denominator, we just sum these impedance products as in equation (12). Note that the impedance of the parallel RC input is R/(1+sCR).

$$V_{out} = \frac{V_1 \cdot R \cdot R + V_2 \cdot R \cdot \frac{R}{1+sCR} + V_3 \cdot \frac{R}{1+sCR} \cdot R}{R \cdot R + R \cdot \frac{R}{1+sCR} + \frac{R}{1+sCR} \cdot R} \qquad (12)$$

$$= \frac{V_1(1+sCR) + V_2 + V_3}{3+sCR} \qquad (13)$$

From this we easily observe that the voltage V_1 is handled differently than the voltages V_2 and V_3. and this we anticipated. The voltage V_1 has a transfer function $T_1(s)$ that is:

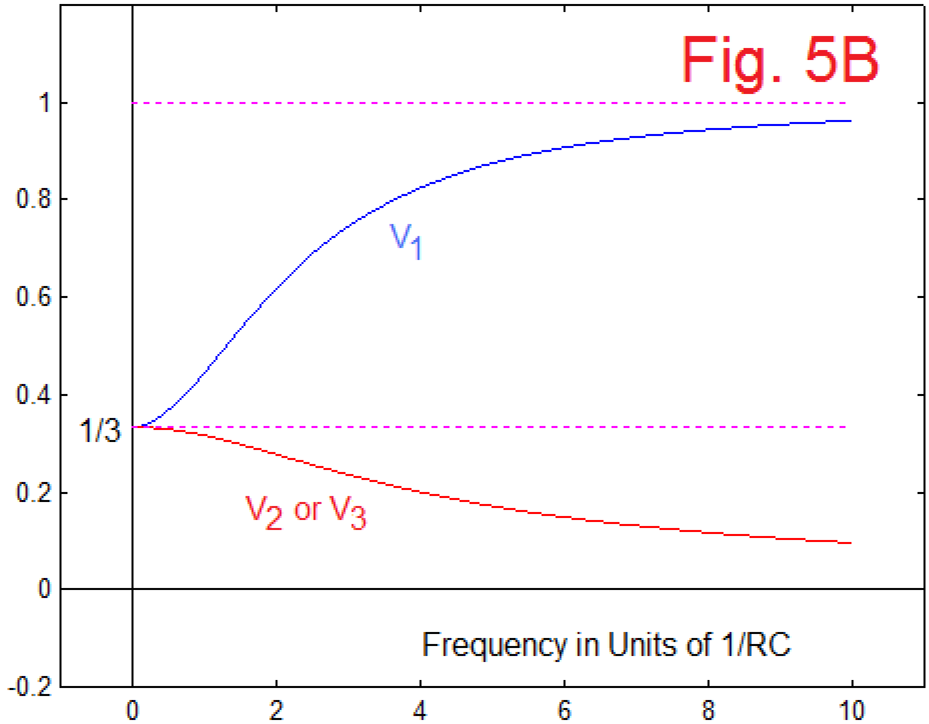
$$T_1(s) = \frac{V_{out}}{V_1} = \frac{1+sCR}{3+sCR} \qquad (14)$$

while V2 has the transfer function:

$$T_2(s) = \frac{V_{out}}{V_2} = \frac{1}{3+sCR} \qquad (15)$$

and $T_3(s)$ is the same as $T_2(s)$.

So we have the answer. Indeed V_1 gets a high frequency boost, but V_2 and V_3 get a high-frequency cut (they are low-passed) by the capacitor in the V_1 leg. Indeed we notice that for high frequencies ($s \to \infty$) $T_1(s)$ goes to 1, while $T_2(s)$ goes to 0. At DC, s=0, both $T_1(s)$ and $T_2(s)$ are 1/3. The corresponding physical interpretations are that in Fig. 5A, at DC the capacitor is effectively thrown out (three equal resistors R) giving 1/3 while at high frequencies, the capacitor C is effectively a short from V_1 directly into the red node. To complete the analysis, the frequency responses corresponding to equations (14) and (15) are plotted in Fig. 5B.

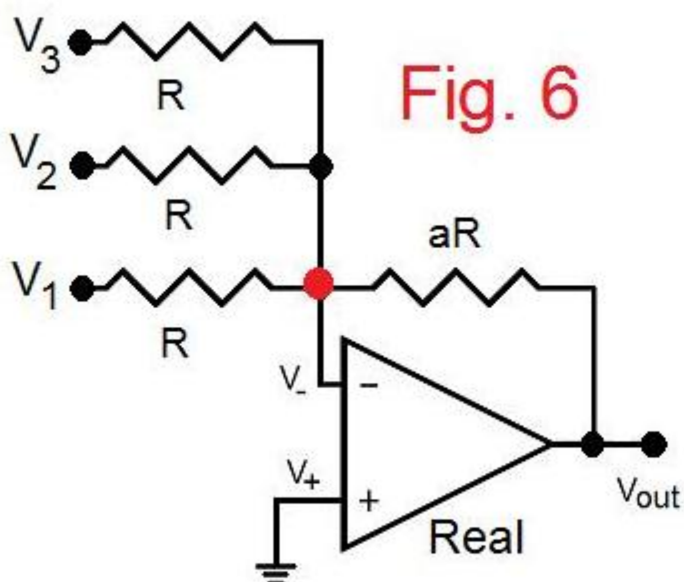


Our second example is the inverting summer with gain a as shown in Fig. 6. The added twist here is that we specify that the op-amp here is real, not ideal. In the ideal case, the (-) input would be a virtual ground, and summing current at the red (summing) node would give us

$$V_{out} = -a(V_1+V_2+V_3) \qquad (16)$$

Here instead we will be using the real op-amp model:

$$V_{out} = (G/s)\,[\,V_+ \;-\; V_-\,] \qquad (17)$$



so while V_+ is clearly 0 (grounded) we need to calculate V_- using the techniques that are the subject of this note. Notice that V_- is NOT just the average of V_1, V_2, and V_3 because we have that nasty resistor aR running to V_{out}. The voltage V_-, the red node, is determined by four impedances to four voltages, V_1, V_2, V_3, and V_{out}.

Again following the verbal recipe, we can write down the equation for the red node of Fig. 6, as marked V_-:

$$V_- = \frac{V_1 \cdot R \cdot R \cdot aR + V_2 \cdot R \cdot R \cdot aR + V_3 \cdot R \cdot R \cdot aR + V_{out} \cdot R \cdot R \cdot R}{R \cdot R \cdot aR + R \cdot R \cdot aR + R \cdot R \cdot aR + R \cdot R \cdot R} \tag{18}$$

Here once again we have a lot of simplification needed, but it is probably a good idea to follow the recipe exactly at first. Our next step is to do this simplification and at the same time to use equation (17) with $V_+=0$, solved for V_- .

$$V_- = \frac{a(V_1 + V_2 + V_3) + V_{out}}{1 + 3a} = \frac{-s}{G} V_{out} \tag{19}$$

which is then rearranged as:

$$V_{out}[\ (1 + 3a)s + G\ ] = -aG\ (\ V_1 + V_2 + V_3\ ) \tag{20}$$

or then:

$$T(s) = \frac{V_{out}}{V_1 + V_2 + V_3} = \frac{\frac{-aG}{1+3a}}{s + \frac{G}{1+3a}} \tag{21}$$

Note that this is –a as G→∞ and/or as s→0. But the important result is that the summer has a pole at –G/(1+3a). Accordingly, it is a low-pass filter. Things should be fine as long as we don't try for frequencies approaching G/(1+3a).