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APPLICATION NOTE NO. 389

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FUN WITH PROBABILITY PUZZLES 1 –
THE MONTY HALL PROBLEM

Possibly one of the most famous problems involving an apparent paradox of probability notions is the so-called “Let’s Make a Deal” or “Monty Hall” or “Marilyn vos Savant” problem. This is derived from a TV quiz show, which I admit I have never seen, but the details of its reality are not important. Most famously, Marilyn vos Savant gave this problem in a 1990 article in ***Parade*** magazine. She gave the correct answer (she is supposed to have the highest recorded IQ of anyone in the world, after all!) and was attacked by many “experts” who told her she was wrong (saying such things as “I have a PhD in math and assure you that you are completely wrong.”).

The problem is counterintuitive to be sure. The usual explanation is convincing, but leaves you as less than totally comfortable in that when you go through it, you are convinced, but then reiterating it to someone else does not seem that transparant. Here I will present the standard explanation and then two additional explanations that I find overwhelmingly convincing. The first added explanation comes from expanding the problem so that there are not just three doors to choose from, but perhaps a hundred doors. The second is to run computer simulation of many trials. According to Leonard Mlodinow in his 2008 book ***The Drunkard’s Walk***, the great mathematician Paul Erdös denied the correct answer, even when given a formal proof, but did succumb to the computer simulation. It is amusing that Erdös who was so much in love with proofs (see ***The Man Who Loved Only Numbers*** by Paul Hoffman, 1998), was convinced not by proof* but by an experiment (reality).

So the answer is, as most everyone knows who has ever heard of the problem, YOU SHOULD SWITCH YOUR CHOICE TO THE “OTHER” DOOR. Keep in mind that this does not <u>guarantee</u> you the best prize. Indeed, you may well be giving the best prize away. It’s about the probability of winning during the one chance you do have. Were it the case that you had many chances to play (like the computer simulation), you would have a virtual guarantee of a very favorable result. By the way, what IS the problem?

You are a guest contestant on some TV show where you have the opportunity to win a prize of a new car which is hidden behind one of three doors. Behind the other two doors are goats which you will win if you choose those doors. No one disputes the fact that at this point your chances of winning the car is 1/3, and of getting a goat is 2/3. But now the twist. The show's host **KNOWS** which doors hide the goats and which door hides the car. He could tell you where the car is hidden, but that would spoil the show. But he then does something, and then gives you the opportunity to choose a different door. He has no intention of directly telling you where the car is, but what he does do gives the contestant extra information that can be used to increase the contestant's chances of getting the car (again, in the "long run" of many shows). He opens one of the three doors (of course, not the door you propose to choose), to show that there is a goat behind it. This is very specific: he does not open a door showing the car, nor does he open your choice of door.

So now it seems clear (and is clear) that the car is either behind the door you propose to choose or behind the other closed door. You have the opportunity to switch your choice. Should you?

For certain, if the car is behind the door you first chose, if you switch, you lose the car. When the average person is told that he/she really should switch, this "but what if" question always come forward. The comfortable (erroneous) argument here might be that the car is either behind the door you chose, or the other one, with equal probability. So why change – aren't first guesses supposed to be best? True, you might miss the car by not switching, but isn't that just the same as never choosing right originally? We all make wrong choices, but how much worse, it seems, to have initially chosen right, and then thrown it away.

The simple first correct explanation is that there is of course a 1/3 chance that the car is behind the door you chose, and a 2/3 chance that it is behind one of the other two doors. If you do not switch your choice, you are stuck with this 1/3 probability. You just ignore the host's shenanigans.

On the other hand, the host does give you additional information – he shows you one door behind which the car is not hidden. That leaves two doors, and the car must be behind one of these two. The probability is 1.0 that it is behind one or the other. Since the probability is 1/3 that it is behind the door you chose first, it must be 2/3 that it is behind the remaining door. This is apparently correct. There are many many elaborations with good illustrations, supporting this general argument.

This contrasts with the incorrect view that you have the choice of two equally likely doors. Curiously, the puzzle is usually presented in terms as to whether you should switch (to improve your chances) or should stay with your first choice (with unspecified consequences). No one really argues that you are better off staying with your first choice, except for the "first choices are best" presumption. So, if you were really a contestant on the show, the answer to whether or not you should switch is YES. This is simply because you have heard of people who have studied the issue who say that it **does** improve your chances (whether you understand the math or not), while others argue from "common sense" that it makes **no** difference.

So at this point you are quite possibly convinced that it improves your odds to switch. Here we take up the first of our two additional demonstrations. Suppose that there are not three doors but 100 doors (one car, 99 goats). If you choose a particular door, you have very little chance, 1 in 100, of having the car, and 99 chances in 100 that the car is elsewhere. Now, the host opens 98 doors revealing 98 goats. This is a huge amount of additional information. You also know that the host knows where the car is, and it is either behind the door you chose (1 in 100), or behind the one of the 99 others that he did not open (99 in 100). Of course you should switch from the almost hopeless case to the almost certain case. The math is the same with three doors, just less dramatic.

The second of our two additional demonstrations is a simulation. We will do this by computer simulation, but you could well imagine inveigling a bunch of kids into spending an afternoon play-acting and keeping score. With perhaps as few as 20 games, evidence would likely show the advantage of switching. Martin Gardner and others have suggested simulations with props and other methods. We use a computer.

Here the Matlab program ***goat.m*** is used to run 10 million games. This is a simple-minded program that just acts-out the game scenario with ***if*** statements. The variable **car** (1, 2, or 3) is the door that hides the car, **choice** (1, 2, or 3) is the chosen door, and **open** (1, 2, or 3) is the door the host opens. The function ***rand*** is just a random number uniformly distributed from from 0 to 1. The variable **getcar** is the "no-choice" sum of successes, while **getcarstay** and **getcarswitch** represent the number of successes for staying with the first choice, and switching, respectively.

Here is the output, the like-notated probabilities:

```
N =    10000000
prob =          0.3335
probstay =      0.3333
probswitch =    0.6667
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PROGRAM GOAT.m

```
% goat.m
% test of "Monty Hall" problem

% first test setup without opportunity to switch
getcar=0;
N=10000000
for k=1:N
   car=ceil(3*rand);
   choice=ceil(3*rand);
   if choice==car;getcar=getcar+1;end
end
prob=getcar/N
% end of no opportunity to switch runs

% new set up the same thing
getcarstay=0;
getcarswitch=0;
for k=1:N
   car=ceil(3*rand);
   choice=ceil(3*rand);
       % host sees contestant chose right door (three cases) - open one of goats
   if choice==1;
      if car==1;
          if rand>.5; open=2; else open=3; end
       end;
   end
   if choice==2;
      if car==2;
          if rand>.5; open=1; else open=3; end
       end;
    end
   if choice==3;
      if car==3;
          if rand>.5; open=1; else open=2; end
       end;
   end
```

```
% OR - host seen contestant NOT chose right door - door has goat
%     most open remaining goat door
 if choice==1;  if car==2; open=3;  end; end
 if choice==1;  if car==3; open=2;  end; end
 if choice==2;  if car==1, open=3;  end; end
 if choice==2;  if car==3; open=1;  end; end
 if choice==3;  if car==1; open=2;  end; end
 if choice==3;  if car==2; open=1;  end; end

 % contestant stays with choice
 if choice==car; getcarstay=getcarstay+1;end

%  OR -  changes choice to final choice, fchoice
if choice==1; if open==2; fchoice=3; end; end
if choice==1; if open==3; fchoice=2; end; end
if choice==2; if open==1; fchoice=3; end; end
if choice==2; if open==3; fchoice=1; end; end
if choice==3; if open==2; fchoice=1; end; end
if choice==3; if open==1; fchoice=2; end; end
if fchoice==car;  getcarswitch=getcarswitch+1; end

end  % end for loop

probstay=getcarstay/N
probswitch=getcarswitch/N
```

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That pretty well answers the question. The probability of getting the car doubles from 1/3 to 2/3 by switching. Note that this as a form of "conditional probability" because we want to know the probability that the car is behind the door you switch to given that it is not behind a door you have had opened for you. So this would also seem to be a problem to which we can apply Bayes Theorem, and this will be the subject of an upcoming app note.

** In Hoffman's book, he tells how Erdös traveled the world, more or less appropriating the hospitality of one potential collaborator after another, living out of a battered suitcase, making noise early in the morning until he woke his host up, so that they could begin proving theorems!*