ELECTRONOTES
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APPLICATION NOTE NO. 386

October 1, 2012

FUN WITH A ROUND EARTH

INTRODUCTION:

All good scientific evidence indicates that the earth we live on is round; not flat. Anyone who denies this would almost certainly have to be judged willfully obtuse, not merely ignorant. Nonetheless the characterization "flat-earther" is often used by many as an invective for a person they perceive to be opposed to a non-mainstream view on a scientific issue. Such a mainstream view is often motivated by a notion of certainty that is politically or popularly expressed, rather than scientifically supported, even in cases where the fundamental scientific and analytic skills (and level of study) of the person under attack exceed those of the attacker by orders of magnitude.

What better way to counter a flat-earther attack than to express in detail how one calculates the declination of the surface as a function of distance away, in terms of trig, geometry, radians, small angle approximation, etc., and then ask the attacker if you have done that analysis correctly. Well, that opportunity is unlikely to happen. So why don't we look at this just for fun. Likely every reader here knows pretty much how to do this. You likely also recognize that as you go further and further away, the rate of decline increases until it becomes infinite at 1/4 of the earth's circumference away (and you fall off the edge – sorry – couldn't resist).

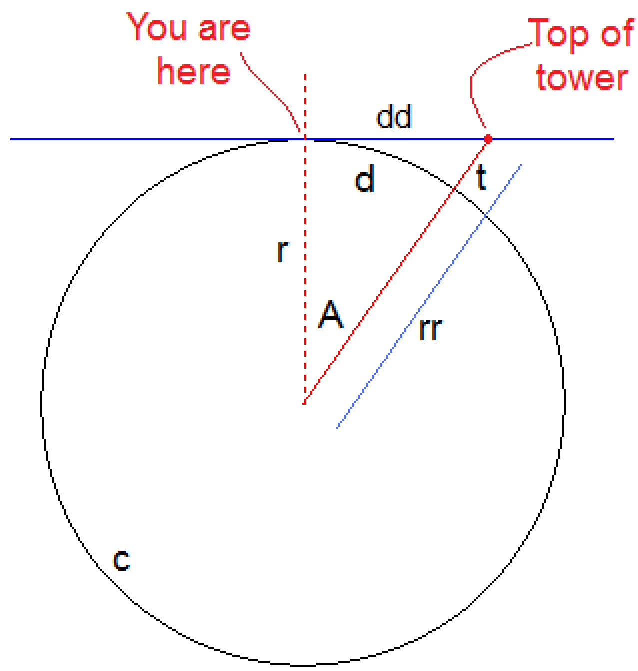
For the moment, check your intuition. Say you go 1 mile away, 10 miles away, 100 miles away, and 1000 miles away. Guess, without doing any calculations, how far below the horizontal plain where you started you now are. Put another way, say you travel this distance and are required to then build a tower to see all the way home. How tall would it have to be for those four distances? Answers at end of this note.

Assuming you did make a guess, look at the answers at the end, and were perhaps surprised, one thing to note is that the correct answers are expressed there in different units, inches, feet, and miles. This is kind of necessary, or else we are delayed by having to convert to units more immediately familiar. So we keep this in mind for our simple programs. Secondly, it's of course not linear, far from it, but you knew that.

PROBLEM 1 - THE DECLINE:

There is really only one geometry problem to solve here, and this is diagrammed in the figure at the right.

Here for convenience we assume you are at the top of the earth. Looking out you really might suppose the earth was flat, as shown by the blue line tangent to the earth's surface. Consider a distance d <u>along the curve</u> away from you. The angle A is:



$$A = (d/c)(2\pi) = d/r \qquad (1)$$

where A is in radians, and c is the circumference of the earth ($2\pi r$), where r is the radius. The radius r is roughly 4000 miles, with 3957 a reasonable average values). Then knowing the angle A, the distance dd is:

$$dd = r \operatorname{Tan}(A) \qquad (2)$$

and this gives us the total hypotenuse (rr) as (Pythagoras):

$$rr = (r^2 + dd^2)^{1/2} \qquad (3)$$

So t, the required height of the tower, is just:

$$t = rr - r \qquad (4)$$

and that's the answer. So that's one application.

PROBLEM 2 - THE HORIZON:

The exact same figure and math applies to the reverse problem – a second application. If you are at an elevation above the surface of the earth, how far away is the horizon? In this case, we reverse the calculations. We start with t – perhaps now t stands for "tall".

$$rr = r + t \tag{5}$$

$$dd = (rr^2 - r^2)^{1/2} \tag{6}$$

$$A = \mathrm{Tan}^{-1}(dd/r) \tag{7}$$

$$d = cA/2\pi = rA \tag{8}$$

PROGRAMS:

The two sets of equations (1)-(4) and (5)-(8) describe first the logical mode of analysis, and secondly, the steps needed to code programs that gives the right answers. Two very simple Matlab programs are shown below. The program ***earthdecline.m*** inputs a distance from a starting point and tells us how tall a tower at that displaced point would have to be to see back to the start. This assumes the tower is to be constructed vertically at the distant point. [If we were allowed to tip the tower back by an angle A, it could be shorter by a factor Cos(A)! So, that's a different answer – the answer to how low below your horizontal plane (perpendicular to it) the land is at that distant point, as opposed to how high up you would need to go vertically from that distant point to reach the plane. Worth keeping in mind if you want to be a pest!] Note that the input is in miles, the output is given in miles, feet, and inches so that you can use whichever is most convenient.

```
% earthdecline.m

% How far below level is a point on the earth's surface
%  a distance d away where d is in miles.
function [miles,feet,inches]=earthdecline(d)

r=3957;
c=2*pi*r;
A=(d/c)*2*pi;       % angle in radians

dd=r*tan(A);
rr=sqrt(r^2+dd^2);
miles=rr-r;
feet=5280*miles;
inches=feet*12;
```

The second program, ***horizondist.m***, is essentially the question of how far you can see. Here you input your elevation above the surface, and ask how far away the horizon is. This might be your height, the height of a building, or the altitude of an aircraft. All of these we generally think of as specified in feet (even aircraft altitudes), while the output gives the distance to the horizon in miles. These seem like the most familiar units.

```
% horizondist.m
%  How far away is the horizon (in miles) for
%    a height t in feet
function miles=horizondist(t)
r=3957;
c=2*pi*r;

rr=t/5280+r;
dd = sqrt(rr^2 - r^2);
A=atan(dd/r);
d=c*A/(2*pi);
miles=d;
```

Clearly these programs give fun and interesting answers. Practical matters such as towers that are miles high (let alone tipping backward!) as well as questions as to whether you can see the horizon excepts as mountains and tree lines, are almost certain complications.

So, while there are practical issues, we can note that there are useful applications of these ideas. For example, we are probably never actually at the "earth's surface" (like sea level), but it does tell us decline for such things as siting of communications towers. For example, a tower 30 miles away would start out about 600 feet below where you read it from a topo map. On the other hand, that mountain 20 miles away that you worried was in the way is 267 feet lower than you read from the map. And so on.

FULL EQUATIONS, LIMITS, SMALL-ANGLE APPROXIMATIONS:

The sequence of equations above, and the corresponding code does just fine and likely adds much insight relative to a single equation. Yet it will be useful to write the results as single equations for the purpose of looking at the results in the limits, and for approximations. So we can combine the stepped equations for the decline together:

$$t = r \{ [1 + \mathrm{Tan}^2(d/r)]^{1/2} - 1 \} \tag{4a}$$

In the limit of $d \rightarrow 2\pi r/4$ (1/4 of the way around), d/r goes to $\pi/2$ (90°) and $t \rightarrow \infty$. This means that the tower needs to be infinite, and this is correct. The tower pointing straight up from the surface would be parallel to the starting plane*. The limit of d=0 is of course, t=0. Don't go anywhere and you don't need any tower.

The corresponding equation for the horizon (setting t to miles) would be:

$$d = r \, \mathrm{Tan}^{-1}[(t^2 + 2rt)^{1/2} / r] \tag{8a}$$

This has some interesting limits. At t=0, d=0. When you are on the ground, you don't see anything! What happens when $t \rightarrow \infty$. We need the inverse tangent of ∞, which is $\pi/2$. Thus $d = r(\pi/2) = c/4$, which is 1/4 of the way around the earth. What this means is that if you are really really tall, get far enough from the earth, you can see half of it. Of course.

Note that equation (1), A=d/r, is true <u>for any angle</u>, not just small angles. This is because we are going round the circle. In a small angle approximation, not only is this the angle, but also the tangent. (It is also the sine in the small angle approximation – recall the "SRT" scale on old-fashioned slide rules) We can say that $\mathrm{Tan}(A) \approx A$ and thus $\mathrm{Tan}^{-1}(A) \approx A$. Making this approximation in equation (4a) we have:

$$t = r \{ [1 + (d/r)^2]^{1/2} - 1 \} = (r^2 + d^2)^{1/2} - r \tag{4b}$$

The corresponding equation for d starts with d=rA and $A = \mathrm{Tan}^{-1}(dd/r)$, so for the small angle d=dd. Thus:

$$d = (t^2 + 2rt)^{1/2} \tag{8b}$$

These two small-angle approximations, equation (4b) and equation (8b) are probably not very useful. They do avoid the Tan and Tan^{-1}, but most program languages have these built in, and there would always be a lagging doubt as to the amount of error that remains.

<u>PROBLEM 3 – HIGH POINT TO HIGH POINT</u>:

There is one remaining problem of practical importance which it may seem we have not solved. This is the problem of two towers. If we have one tower or height t_1 and a second of height t_2, how far apart can they be and still see each other's tops? That is, we move the towers apart until a straight line from the top of one to the top of the other is just tangent to the earth's surface. This is not a separate problem from the horizon problem – we can just use it twice – once for each tower and combine the distances. For example, if we had a 300 foot tower (horizon at 21 miles) and a 500 foot tower (horizon at 27 miles) they could see each other at a total separation of 48 miles. At the point 21 miles from the first tower, in theory, a signal would skim across the ground and continue on 27 miles to the second tower.

Of course you might worry: what if there is a 100 foot mountain there. Perhaps. We can add to the problem that the first tower might well be on a 200 foot mountain itself, and the second one on a 300 foot mountain. This just changes the effective heights of the towers. The whole question of whether one tower is actually in sight of the other can be complicated. The useful trick here is to then consider an <u>imaginary earth's surface</u> that is a "shell" at the elevation of the top of the mountain in between. If the "horizon" for the first mountaintop tower to this <u>elevated surface</u> meets or is beyond the mountain, then the signal clears the mountain. The same must be true of the second mountaintop tower.

<u>ANSWERS FROM PAGE 1</u>

1 mile → 8 inches

10 miles → 67 feet

100 miles → 1.26 miles

1000 miles → 130 miles

* NOTE FROM TEXT: As we stated, there was a difference between the "vertical construction" notion of how far below the original plane the distant point is, and the minimum distance from the plane perpendicular down to the distant point. This was a matter of adding a factor Cos(A) to equation (4a) as below:

$$t = r \,\mathrm{Cos}(d/r) \{ [1 + \mathrm{Tan}^2(d/r)]^{1/2} - 1 \}$$

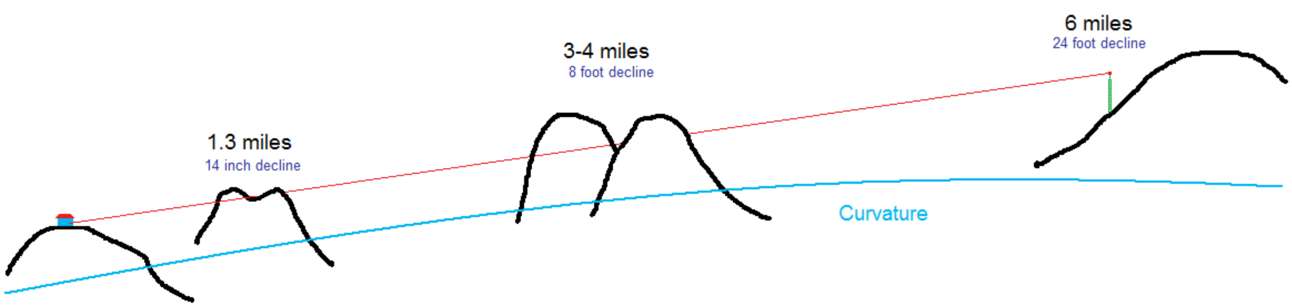
Here in the limit where $d \to 2\pi r/4$, d/r goes to $\pi/2$ (90°) and $t \to r$. This is an issue of one limit going to infinity (the tangent) fighting with the limit going to zero (the cosine). This is simply the matter of saying that we would construct a "tower" starting parallel to the ground which would then run 3957 miles to the starting plane. Not too practical!

A Backstory:

Probably 25 years ago, I became interested in this problem of the earth's curvature when visiting my father's house (where I grew up) and he was curious about a newly-seen blinking red light right on the southern horizon. This was up through some hills where the horizon seemed only a couple of miles away (to the north, we could see 30 miles to Rochester). Evidently this was some new tower. It was winter, and the light easily disappeared if one moved just a few feet one way or the other. It had not been seen during the summer, so perhaps it wasn't there then, or perhaps tree leaves were in the way. But also, being winter with snow on the ground, using binoculars it was fairly easy to get a bearing from our house up through the hills, and to get a fairly exact angle. Using topographic maps, we could plot this line. There was nothing there – but towers are certainly erected in remote areas. The nearest feature was an airport, 20 miles away on that line. I calculated the decline at 266 feet at that distance, so that seem unlikely. And that was up and over the hills.

But I knew fairly exactly, almost to the tree, the slight notch on the horizon (about 1.3 miles away) through which the light had to be passing. That notch bottom was declined about 14 inches from the topo map elevation, but that was insignificant. The line was definitely going up. But curiously, the direction of the line placed it to the right of one hill and to the left of another further on in the range of 3-4 miles. The points on the slopes below that line, when adjusted for decline (about 8 feet), were just enough to accommodate the light beam. So what was beyond that? Well, at about 6 miles (24 foot decline) there was the ridge of a gradual upslope of a very large hill, and the line passed <u>over</u> this by some 80 feet. (Further up this hill and to the left, there was plenty of additional elevation, but that was behind the leftside hill back at 3-4 miles.) Beyond this upslope ridge at 6 miles, there was a lake valley for probably another 5-10 miles; nothing high at all.

So a tower of perhaps 100 feet on the upslope ridge was the only possibility. A photon leaving the top of the tower snuck past two slopes at 3-4 miles, one notch at 1.3 miles, and found its way into my father's living room. What are the odds of that? Well, probably pretty good – somewhere.



Oh. Somehow, I didn't get up there to look for the tower until about four years later. It was there, on a new road, and appeared new (the chain-link fence around it was new). It seemed 80-100 feet high.

EXPANDED PROGRAMS

The programs below include the full equations and approximations for reference.

```
% earthdecline.m
% How far below level is a point on the earth's surface
%  a distance d away where d is in miles.
function [miles,feet,inches]=earthdecline(d)

r=3957;
c=2*pi*r;
A=(d/c)*2*pi;        % angle in radians
AngleDegrees=(A/(2*pi))*360

dd=r*tan(A);
rr=sqrt(r^2+dd^2);
miles=rr-r;
feet=5280*miles;
inches=feet*12;

FullEqnMiles=r*((sqrt(1+(tan(d/r))^2)) - 1)
MinMiles=cos(d/r)*r*((sqrt(1+(tan(d/r))^2)) - 1)
SmallAngleApproxMiles=sqrt(r^2+d^2) - r
```

```
% horizondist.m
%  How far away is the horizon (in miles) for
%    a height t in feet
function miles=horizondist(t)
r=3957;
c=2*pi*r;

rr=t/5280+r;
dd = sqrt(rr^2 - r^2);
A=atan(dd/r);
d=c*A/(2*pi);
miles=d;

FullEquationMiles=r*atan(sqrt((t/5280)^2+2*r*(t/5280))/r)
SmallAngleApproxMiles=sqrt((t/5280)^2+2*r*t/5280)
```