## ELECTRONOTES

APPLICATION NOTE NO. 383
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At least it was from Don I saw it first, and probably 20 years ago it was making the rounds (it's a much older problem - perhaps?). Here it is: take a capacitor C, and charge it to a voltage V by whatever means you prefer (Fig. 1a). Now connect it across a second capacitor C (Fig. 1b). No charge escapes, and is thus distributed on both capacitors equally, and the charge and voltage are proportional ( $q=C V$ ), so the voltage goes to $\mathrm{V} / 2$. We envision perhaps moving half the charge in a "bucket" of some sort. Easy enough. Another way of transferring the charge would be to connect them with a resistor (Fig. 1c) and go out to lunch. If lunch time is very long compared to the RC time constant, when we get back we expect equal voltages on both capacitors. Perhaps Fig. 1b is just a special case of Fig. 1c where $R=0$. This is simple.

Fig. 1 Two Capacitors


So here is the problem. First, is charge conserved? Of course, we said that. Is energy conserved? Of course, it always is. So is the energy stored in the capacitors conserved? Well, the energy stored in a capacitor is proportional to $\mathrm{V}^{2}$ : its (1/2) $\mathrm{CV}^{2}$. So in Fig. 1a, setting $\mathrm{V}=1$ and $\mathrm{C}=1$, the energy is $1 / 2$ (just the one capacitor). In Fig. 1 b, there are two capacitors, each with voltages of $1 / 2$, so the total energy is $(1 / 2)(1 / 2)^{2}+(1 / 2)(1 / 2)^{2}=1 / 4$. Half the energy is gone! In Fig. 1c, we also have a total energy of $1 / 4$, but perhaps a clue where the energy went. Was $R$ perhaps hot when we returned from lunch?


Likely most readers can sketch what is going on in Fig. 1c: an exponential "decay" of both capacitors toward the voltage of $1 / 2$, and guessing the right time constants is also fairly easy (see end of note). The curves in Fig. 2 were actually generated in three ways: by guessing, by simulation, and by solving the actual differential equations (below). All three ways gave the same result as we would hope. Note that the procedures here are essentially the ones also used in our recent EN\#210.

## SOLVING THE DIFFERENTIAL EQUATIONS

The circuit with a non-zero resistor $R$ is shown in the upper right of Fig. 2. The current $\mathbf{i}$ is of course just $\left(\mathrm{V}_{1}-\mathrm{V}_{2}\right) / R$. Any dq of charge transferred by this current moves from one capacitor to the other. Hence, we have the two equations:

$$
\begin{align*}
& \mathrm{d} \mathrm{~V}_{1} / \mathrm{dt}=-(1 / \mathrm{C}) \mathrm{dq} / \mathrm{dt}=-(1 / \mathrm{C}) \mathrm{i}=-(1 / R \mathrm{C})\left(\mathrm{V}_{1}-\mathrm{V}_{2}\right)  \tag{1a}\\
& \mathrm{d} \mathrm{~V}_{2} / \mathrm{dt}=(1 / \mathrm{C}) \mathrm{dq} / \mathrm{dt}=(1 / \mathrm{C}) \mathrm{i}=(1 / R \mathrm{C})\left(\mathrm{V}_{1}-\mathrm{V}_{2}\right) \tag{1b}
\end{align*}
$$

The eigen-analysis is shown below:

$$
\begin{align*}
& {\left[\begin{array}{l}
d V_{1} / d t \\
d V_{2} / d t
\end{array}\right]=(1 / R C)\left[\begin{array}{rr}
-1 & 1 \\
1 & -1
\end{array}\right]\left[\begin{array}{l}
V_{1} \\
V_{2}
\end{array}\right]} \tag{2}
\end{align*}
$$

$$
\begin{align*}
& \mathrm{t}=0 \quad \mathrm{t}=0 \quad \text { Initial Conditions } \\
& {\left[\begin{array}{l}
1 \\
0
\end{array}\right]=\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
C_{1} & C_{2} \\
-C_{1} & C_{2}
\end{array}\right]=\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
1 & 1 \\
-1 & 1
\end{array}\right]\left[\begin{array}{l}
C_{1} \\
C_{2}
\end{array}\right]}  \tag{5}\\
& {\left[\begin{array}{l}
C_{1} \\
C_{2}
\end{array}\right]=\left[\begin{array}{cc}
1 / 2 & -1 / 2 \\
1 / 2 & 1 / 2
\end{array}\right]\left[\begin{array}{c}
\sqrt{2} \\
0
\end{array}\right]=\left[\begin{array}{l}
\sqrt{2} / 2 \\
\sqrt{2} / 2
\end{array}\right]} \tag{6}
\end{align*}
$$

So the results really are, plugging the $\mathrm{C}_{1}$ and $\mathrm{C}_{2}$ values into equation (4)

$$
\begin{align*}
& V_{1}(t)=0.5 e^{-2 t}+1 / 2  \tag{7a}\\
& V_{2}(t)=-0.5 e^{-2 t}+1 / 2 \tag{7b}
\end{align*}
$$

So we know what is going on mathematically, and it matches exactly our intuitions. Further we think we understand that energy is not conserved as energy stored in the capacitors because some is converted to heat in the transfer through the resistor. But how much? Is it half regardless of the resistor!

Remember we are looking at the case where the original energy was $1 / 2$, and with the charges equalized on both capacitors, the total energy is quite decidedly now down to exactly $1 / 4$. So we suspect we are loosing $1 / 2$ of $1 / 2=1 / 4$ to heating (or somehow).

It is the total voltage across the resistor that determines the power, and thus the energy, once integrated. This voltage is the difference between the red curve and the blue curve in Fig. 2, and is a pretty ordinary exponential decay to zero (green curve in Fig. 2) as the two capacitor voltages converge. In fact it's $\mathrm{e}^{-2 t}$ for the example. The energy is the integrated power, so we need to calculate the power, which is the voltage times the current. This we have conveniently done by simulation by adding the energy at each of the simulation increments. The program that does the simulation is below. This is convincingly a result that $E=1 / 4$, regardless of $R>0$ ( 0.2501 in the actual example).

## MATLAB PROGRAM - SIMULATION

```
\% AN383.m
\(\mathrm{V} 1=1\)
\(\mathrm{V} 2=0\)
\(\mathrm{C}=1\)
R=3
\(\mathrm{dt}=.001\) \% time increment
\(\mathrm{E}=0 \quad\) \% initialize energy dissipated
kk=1:9000;
for \(k=1: 9000\)
    \(\mathrm{i}=(\mathrm{V} 1-\mathrm{V} 2) / \mathrm{R}\);
    \(d q=i^{*} d t ;\)
    \(\mathrm{V} 1=\mathrm{V} 1-(1 / \mathrm{C})^{*} \mathrm{dq}\);
    \(\mathrm{V} 2=\mathrm{V} 2+(1 / \mathrm{C})^{*} \mathrm{dq} ;\)
    \(\mathrm{V} 1 \mathrm{k}(\mathrm{k})=\mathrm{V} 1\); \% store
    \(\mathrm{V} 2 \mathrm{k}(\mathrm{k})=\mathrm{V} 2\); \(\quad\) \% store
    V1test \((k)=1 / 2+(1 / 2)^{*} \exp \left(-k^{*} d^{*} 2\right)\); \(\quad \%\) results for solving...
    V2test( \(k\) ) \(=1 / 2-(1 / 2)^{*} \exp \left(-k^{*} d t^{*} 2\right)\); \(\%\) differential equations
    \(E=E+i^{\wedge} 2^{*} R^{*} d t ; \quad \%\) increment energy
end
```

AN-383 (4)

```
Vtot=V1k-V2k;
figure(1)
plot([0 8],[.5 .5],'k:')
hold on
plot(kk/1000,V1k,'r')
plot(kk/1000,V2k,'b')
plot(kk/1000,Vtot,'g')
plot([0 8],[0 0],'k')
plot([0 0],[0 2],'k')
%plot(kk/1000,V1test,'g:') % overplot dotted green check
%plot(kk/1000,V2test,'g:') % overplot dotted green check
axis([-.2 8.2 -.1 1.2])
hold off
figure(1)
E
```

Our results here lead to the interesting conclusion that half of the energy stored in the capacitor is dissipated as heat, and that this result is independent of $R$. But the case of $R=0$ (Fig. 1b) is still problematic. In a practical case, there is always some resistance, even some internal to the capacitors. And, in a practical case, we would expect that regardless of V , there is some sort of spark when we initiate contact resulting in heat and light.* People who have discussed this problem (consult the Internet) have also suggested that in the absence of any resistance, an oscillation and resulting radiation would release energy. In fact, sparks do radiate RF as those who played with them while a radio is on, or who messed up TV reception for family members while experimenting, know. Such an oscillation would seem to also occur (damped) with resistance as well. So various means of energy loss may occur simultaneously until the $1 / 4$ excess is used up.

In ones sense, it looks like an incredibly delicate balancing act, a number of energy dissipating mechanisms sharing duty to reach exactly half the original energy. Looking at it from the fundamental conservation laws (energy and charge) there is no mystery about the final result.

[^0]
## PARALLEL, OR SERIES

Earlier we suggested that it easy to intuit that the two capacitors converge exponentially to a voltage of $1 / 2$, and not difficult to guess the right exponential time constant. This is a true statement, allowing for the fact that you are likely to guess wrong first. We know that an exponential decay of an RC circuit should be of the form $e^{-t / R C}$. We also strongly suspect that when the second capacitor is introduced, we are dealing with a factor of two somehow. Probably we would guess that the exponential decay is going to be e ${ }^{-t / 2 R C}$ since the capacitors seem to be in parallel. In my case, I had the simulation running and it was clear that the answer was what would have been the alternative guess: $e^{-2 t / R C}$. Making that guess (figure out WHY later), the simulation, and the differential equation solution, were all in agreement. Now, why is it C/2?

There is something "unfamiliar" about Fig. 1c. That is, once we add the resistor, we are not so obviously just connecting two capacitors in parallel. In fact, if we get rid of the grounds, and just use a wire, as in Fig. 3a, we see that we actually have two capacitors in series, a resulting capacitor $\mathrm{C} / 2$ with a resistor R across it (Fig. 3b). So that's why the decay exponential is $\mathrm{e}^{-2 t / R C}=\mathrm{e}^{-\mathrm{t} /(\mathrm{RC} / 2)}$, and this does look familiar. This explains the observed time constant.


Parallel 2C?


Series C/2

Fig. 3
Is it Really a Series

So, can we solve the original problem with a single capacitor C/2? No we can't. Our final result of two capacitors $C$ in parallel with voltage $\mathrm{V} / 2$ is now, in the series view, two capacitors C with voltage $\mathrm{V} / 2$ back-to-back so that zero net voltage is across the resistor. You can't make this one capacitor. Fig. 3b would discharge completely to zero charge, zero voltage, and zero stored energy. It's not the same.


[^0]:    * Anyone who doubts that a spark from a discharging capacitor can be hot probably is younger than vacuum tubes. With vacuum tubes we often delighted in charging up capacitors through a rectifier tube (a "B Supply") and then using a wire to discharge this through a piece of tinfoil. It melted holes in the foil.

