ELECTRONOTES
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APPLICATION NOTE NO. 375

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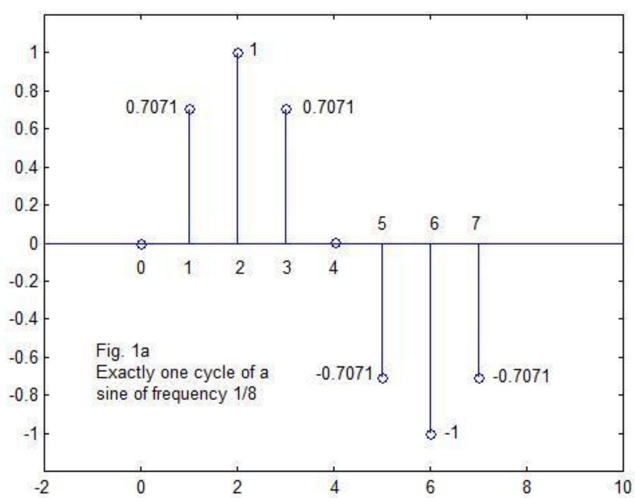
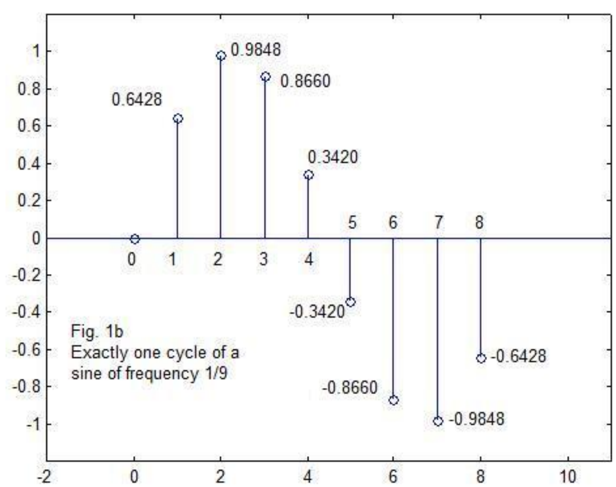
MOVING AVERAGES AND SINGLE SINEWAVE CYCLES

Moving average filters have a well known frequency response in the form of a periodic sinc, so we know what frequencies are to be nulled out. Further, single cycles (or an integer multiple of cycles) of a sinusoidal waveform have samples summing to zero. Hence they all are nulled in any average. However, single cycles can be nulled in cases where a corresponding periodic waveform is not. This leads to some interesting insights.

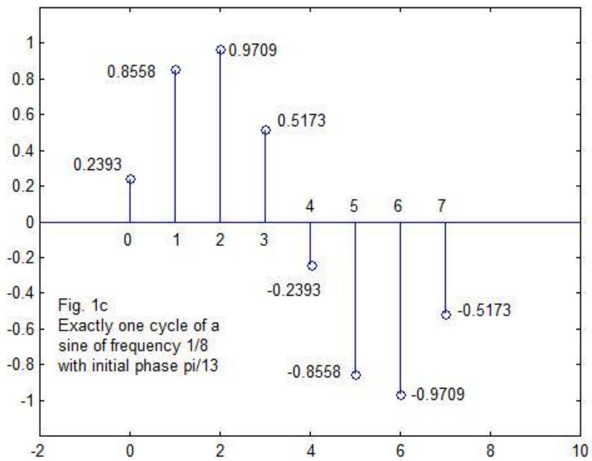
SINGLE CYCLES:

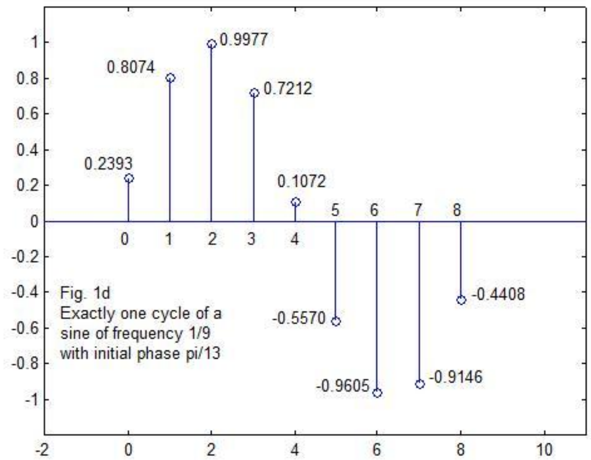
Consider a sequence of numbers that corresponds exactly to one full cycle of a sinusoidal waveform. What is the sum of these samples? If there are an even number of samples, it is obvious that the sum is zero (Fig. 1a), as for each positive sample there is a corresponding negative sample. If we move to an odd number of samples, this is still true since the first sample is zero and the remaining eight balance (Fig. 1b).





It remains true but less obvious for an even number of samples but with a non-zero starting phase (chosen as π/13 here – Fig. 1c). For the case of an odd number of





samples with a starting phase (chosen as π/13 here – Fig. 1d), it is still true but far from obvious and we don't see pairwise cancellations. It is easy to run many computer cases to convince yourself that it is <u>true</u>. Something like 1000 Matlab examples of

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s=sin(2*pi*[0:(N-1)]/N + ph)
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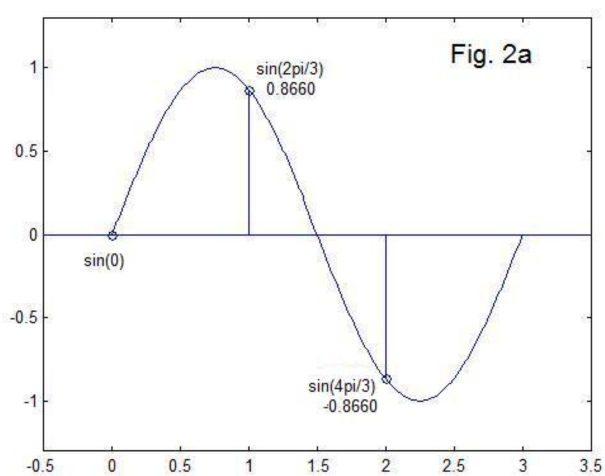
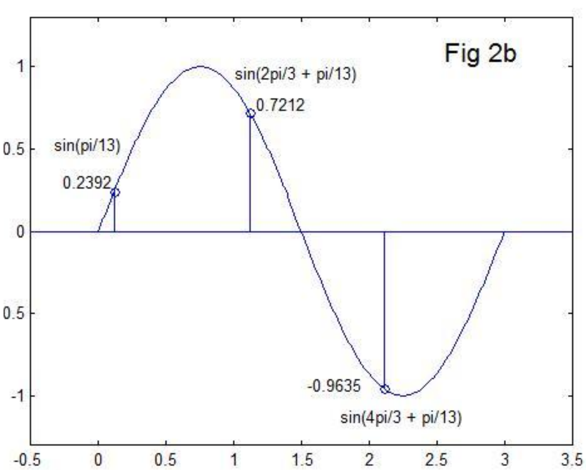
followed by sum(s) with random integer N and random phase ph should convince us.

In order to gain some notion of how the most dubious case (odd number of samples and a non-zero phase) sums to zero, we can use trig identities for a length 3 case. For the example of Fig. 2, we choose N=3 and ph=π/13. In Fig. 2a we have the case of a length





3 sinewave where the phase starts at zero, and the fact that the three samples add to zero is obvious from the symmetry. In Fig. 2b, we have chosen a non-zero phase, and symmetry is lost.

For the general case of exactly three samples per cycle and a phase angle α, we want to sum:

$$S = \sin(\alpha) + \sin(2\pi/3 + \alpha) + \sin(4\pi/3 + \alpha) \qquad (1)$$

We can use the trig identity for the sine of a sum:

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y) \qquad (2)$$

and we thus get a total of five terms:

$$S = \sin(\alpha) + \sin(2\pi/3)\cos(\alpha) + \cos(2\pi/3)\sin(\alpha) + \sin(4\pi/3)\cos(\alpha) + \cos(4\pi/3)\sin(\alpha) \qquad (3)$$

But since $\sin(2\pi/3) = -\sin(4\pi/3)$, (as in Fig. 2a for example) two of the terms cancel:

$$S = \sin(\alpha) + \cos(2\pi/3)\sin(\alpha) + \cos(4\pi/3)\sin(\alpha) \qquad (4)$$

And further since $\cos(2\pi/3) = \cos(4\pi/3) = -0.5$, we get:

$$S = \sin(\alpha)\ [\, 1 + 2\cos(2\pi/3) \,] = 0 \qquad (5)$$

which shows the sum for three samples per cycle is zero regardless of the choice of α.

While not a formal proof, we feel we have established, through several avenues, that the sum (and average) of samples taken this way is always zero.

A MOVING AVERAGE FILTER

A moving average filter takes the sum of N consecutive inputs and divides by N. We are concerned here with when this average is zero and when it is not, so we will be content to take either the sum or the average. The frequency response for the moving average filter is given by (see EN#197 (6) for example):

$$H(f) = (1/N)\, e^{-j(N-1)\pi f}\ [\, \sin(N\pi f) / \sin(\pi f) \,] \qquad (6)$$

which is the familiar Dirichlet function on "periodic sinc". We note that the function has zeros when $f = m/N$ [for $\sin(N\pi f) = 0$] for m not zero or an integer multiple of N. Accordingly, we think we understand that these frequencies are rejected by a moving average filter. On the other hand, we arguing that single cycles will be rejected for any length moving average (easily seen as long as the length is perhaps several times longer than the length of the single cycle input so that we can see the zone of zeros in the middle of the output). Accordingly there are two ways a particular frequency (or apparent frequency) can be rejected. These cases will be illustrated below.

TEST CASES

Here we will employ three different inputs, and two moving average filters. The inputs are:

s1 12 samples of a sinusoidal waveform of frequency 1/12=0.08333…(one full cycle)

s2 s1 repeated 10 times, 10 full cycles, 120 samples

s3 120 samples of a sinusoidal waveform of frequency 5/51= 0.0980…..

And we will have moving average filters of length 48 (h1), and of length 51 (h2). For the experiment, we will use the moving sum rather than the average.

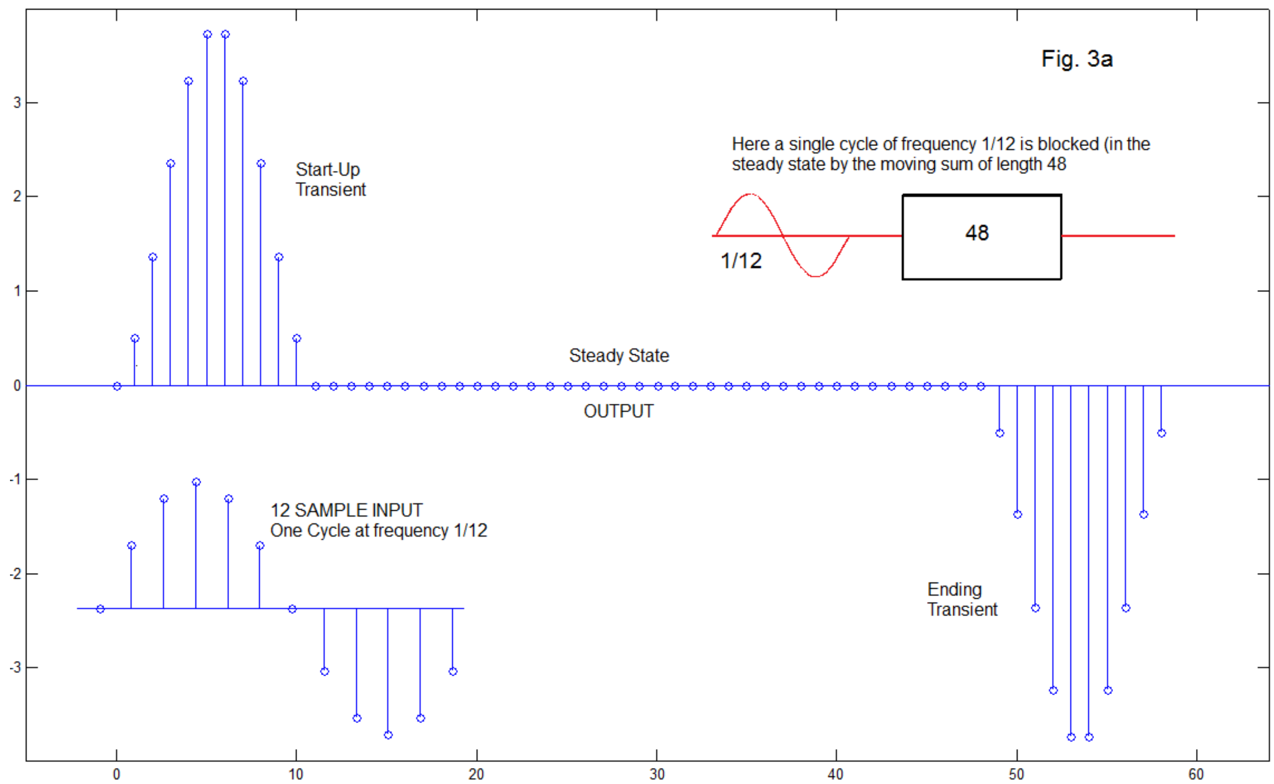
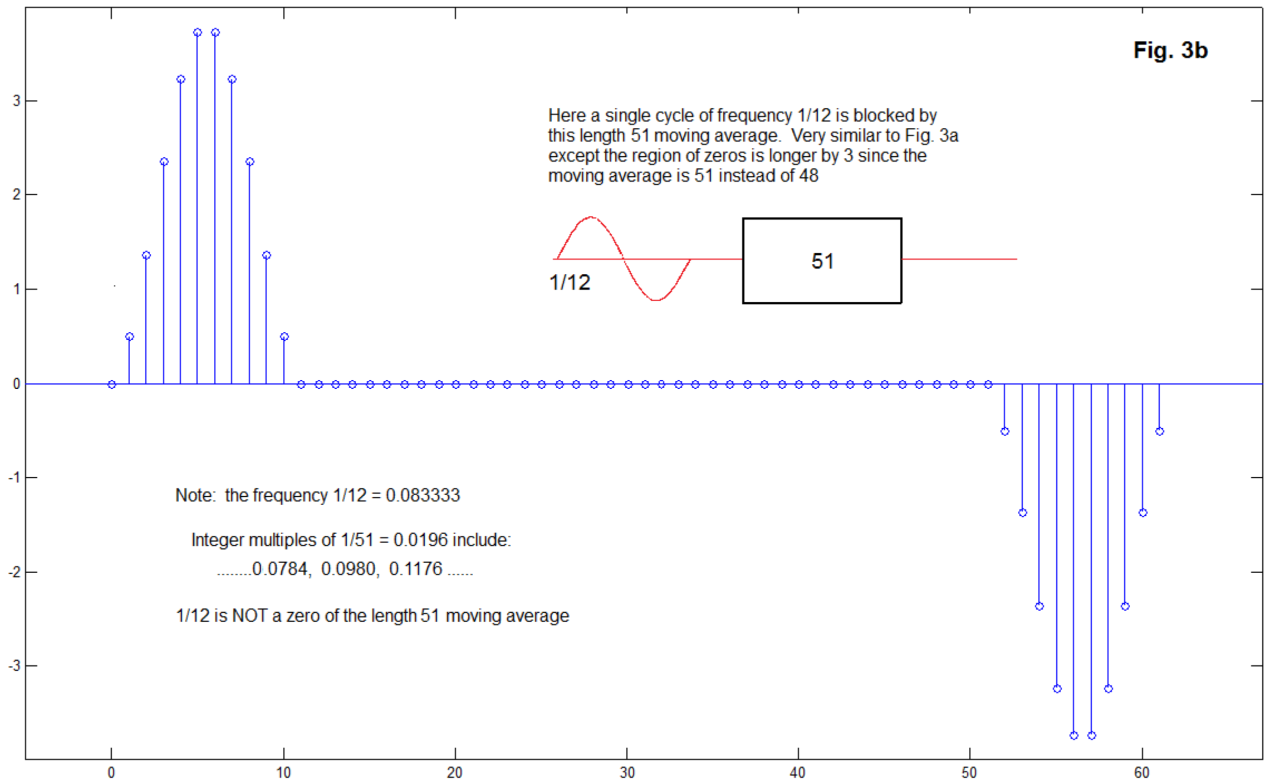
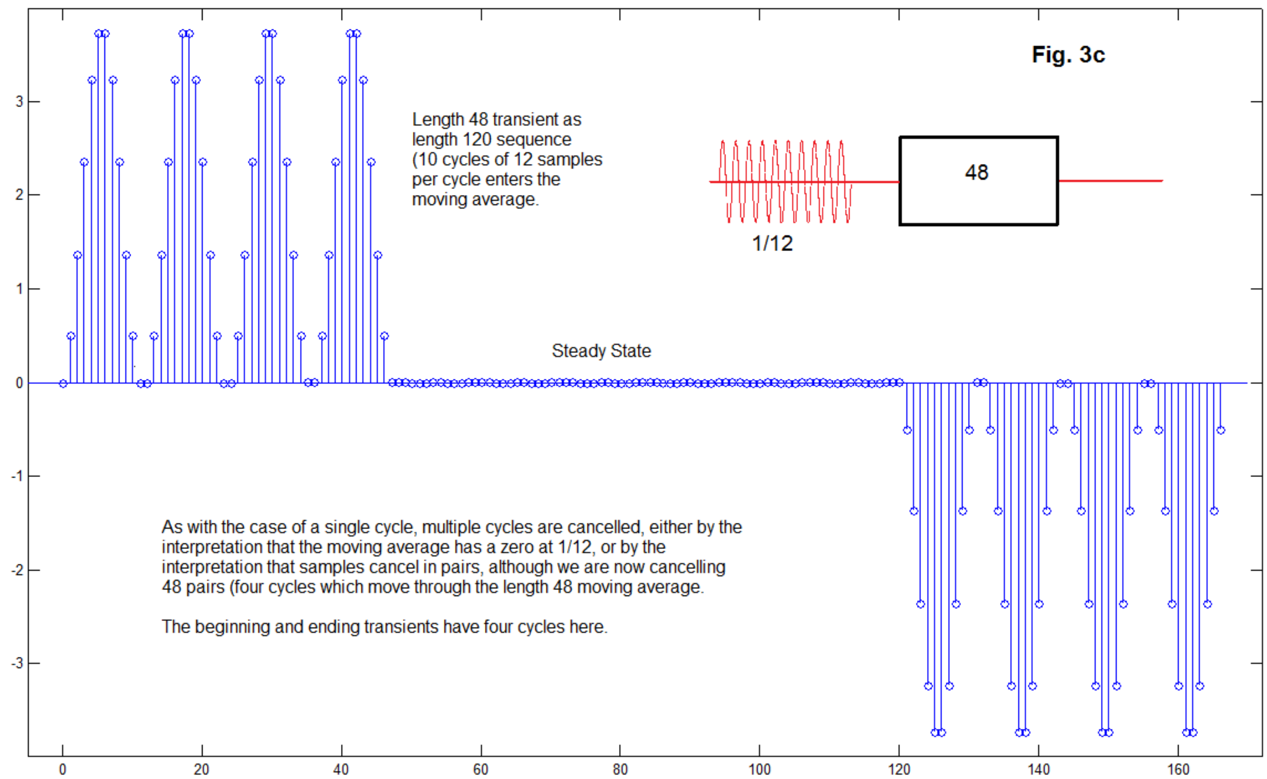


Fig. 3a shows the output for the case of the single cycle, s1, and the length 48 moving sum, h1. The output is the convolution of the two. We note a length 12 beginning transient and a length 12 ending transient, as expected. Recall the s1 is time reversed in the convolution. In between the transients we get all zeros as we anticipated. So a moving average does reject a single cycle. It is true here that the frequency 1/12=0.083333… is the fourth zero of the frequency response of the length 48 moving average. But see the next case.



In Fig. 3b, we have a case that is apparently very similar to Fig. 3a. Here the moving average is length 51 and accordingly it's zeros are integer multiples of 1/51. Clearly1/12 is not an integer multiple of 1/51. The transients are the same, and the middle region is zero. So still the single cycle of 1/12 is rejected, and we clearly understand this as samples cancelling each other. Clearly the zeros of the moving average are only part of the story.

In the next test case, shown in Fig. 3c, we use the s2 input and the h1 filter. That is the input becomes a full ten cycles of the sinewave of frequency 1/12, or 120 samples total, somewhat longer than the filter itself. Here we are trying to see what happens when we



think of the input as not a short event (one cycle) but rather more of a steady state situation. The result is curious. First note that we do have the zero steady state between two more extended transients. Here it is still possible to see how 48 samples total (four cycles) always cancel in pairs, just as we saw how 12 samples (one cycle) did. It is also true that the moving average filter is not influenced in this center (steady state) region by the finite length (120 samples) of s2. For this region, it might as well be the case that the sinusoidal goes on forever. Accordingly, we appreciate the null of the moving average at 1/12. We note that the transients are four time linger than the original case. We see four "cycles" to the transient as it takes 48 samples (four cycles at 1/12) to "load" the filter.

In our three examples so far we have not seen anything, other than transients, get through the filter. However if we take the length 51 moving average and put the length 120 sinusoidal of frequency 1/12, something should come through. Fig. 3d shows this case. The transients look similar (slightly longer) to Fig. 3c. But indeed there is a significant response in the middle. Why? First, there is no way 51 samples of a sine periodic with period 12 can cancel. Secondly, the moving average itself has no zeros at 1/12.

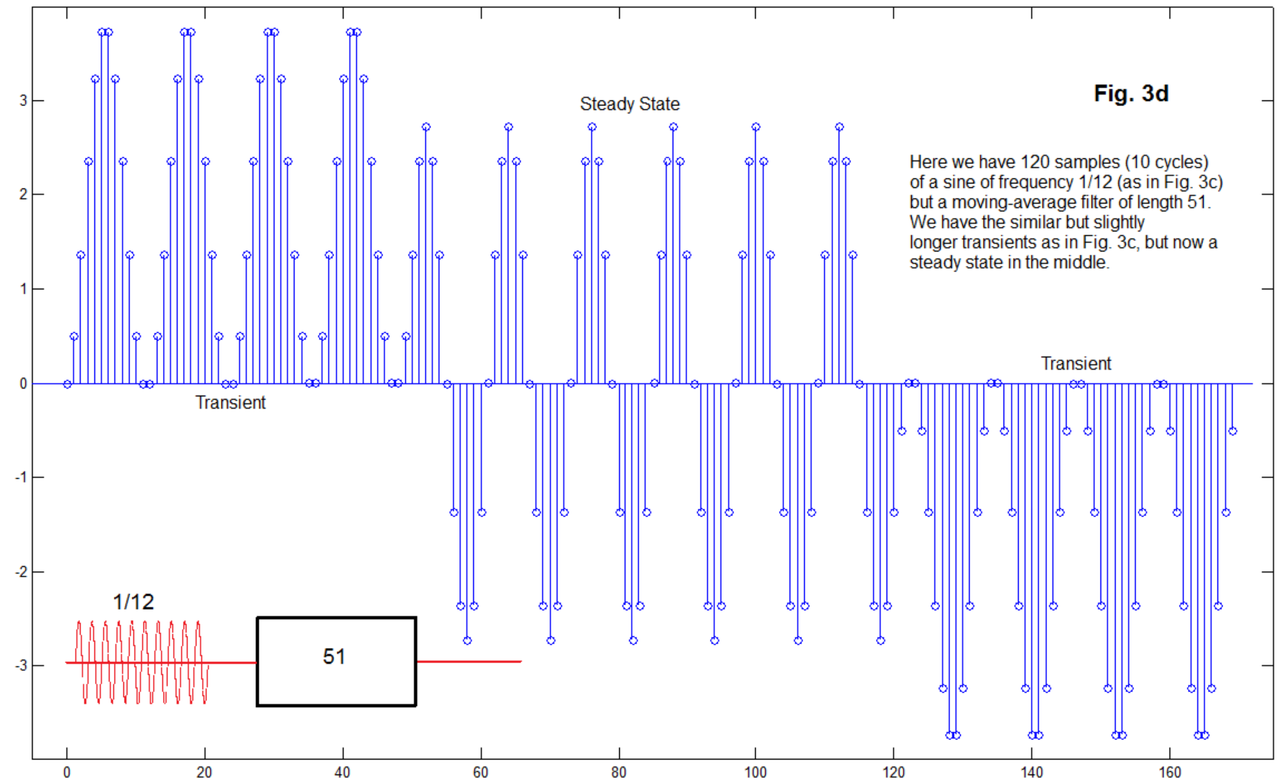


Fig. 3d
Steady State
Transient
Transient
Here we have 120 samples (10 cycles) of a sine of frequency 1/12 (as in Fig. 3c) but a moving-average filter of length 51. We have the similar but slightly longer transients as in Fig. 3c, but now a steady state in the middle.
1/12
51

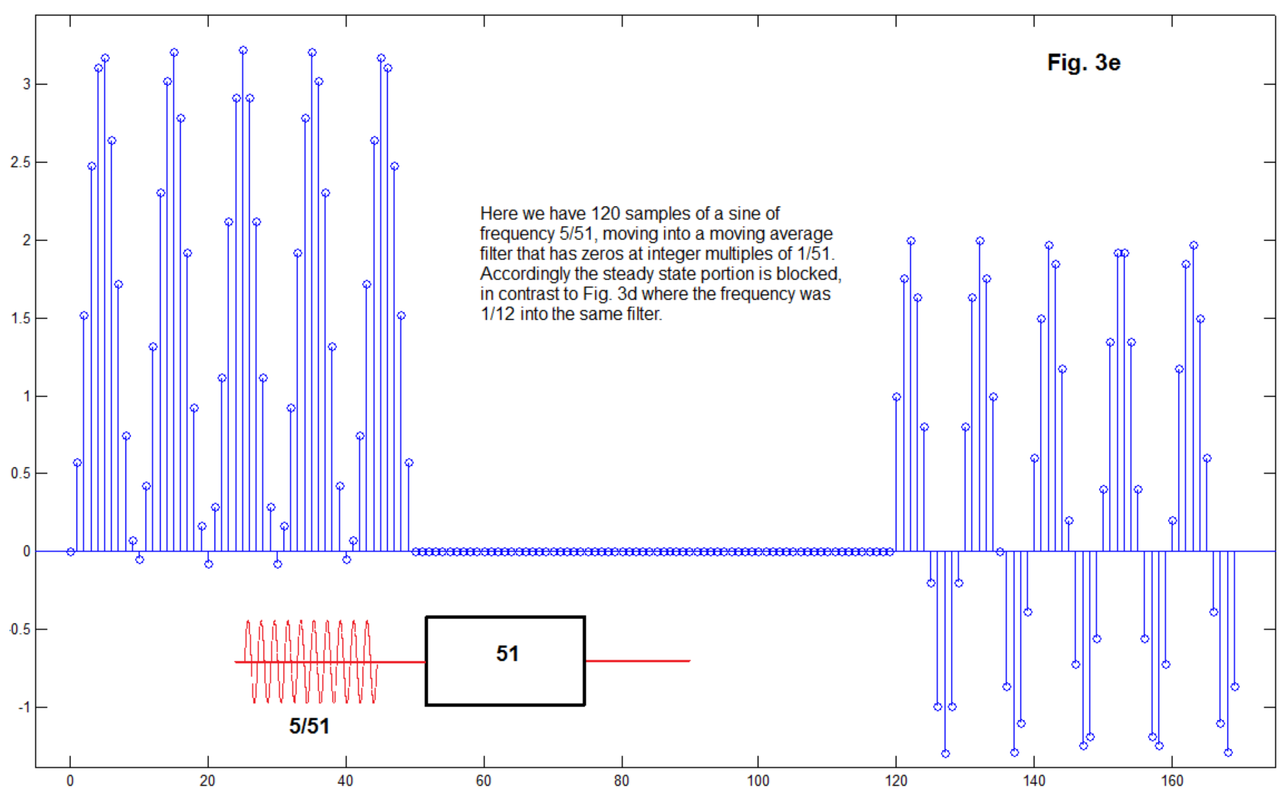


Fig. 3e
Here we have 120 samples of a sine of frequency 5/51, moving into a moving average filter that has zeros at integer multiples of 1/51. Accordingly the steady state portion is blocked, in contrast to Fig. 3d where the frequency was 1/12 into the same filter.
51
5/51

ZEROS OF THE INPUT!

The cases seen in Fig. 3c, Fig. 3d, and Fig. 3e present our usual notion of a filter with zeros at certain frequencies blocking these same frequencies while letting other frequencies pass. Here while the input sequences are not of infinite length (to be considered pure frequencies) they are much longer than the filter's storage and thus it makes no difference.

So the special cases here are Fig. 3a and Fig. 3b. We have argued from time-domain considerations that full cycles of any sine wave should cancel in a moving average, and they do. Accordingly, while the moving average filter has zeros for certain frequencies, the frequency of any sinewave cycle is irrelevant. Indeed, it does not even have to be a sinewave of course. Any sequence that sums to zero, and which fits within the length of the moving average, will give a zero steady state.

This view tends to be the standard one of considering the (longer length) "moving average filter" to be the filter in this case, with the (shorter length) "input" being the input. In as much as the output here is the convolution of two sequences, and we know that the convolution can be done in either order, we can consider the situation to be sine wave samples (or whatever alternative balanced choice we make) as the filter with an input of a train of constant sample (Fig. 4).

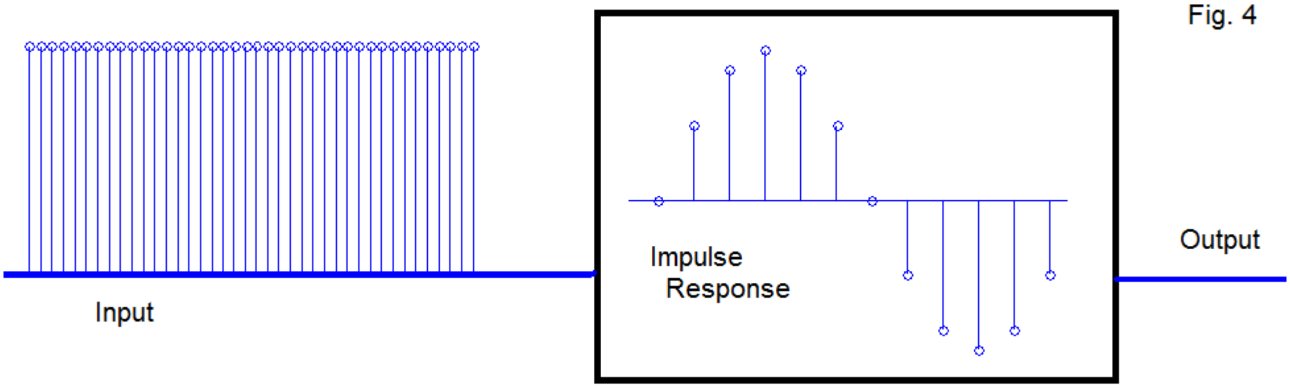
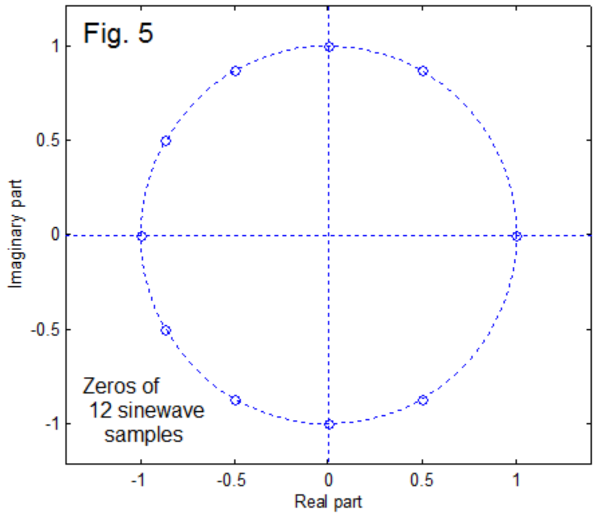
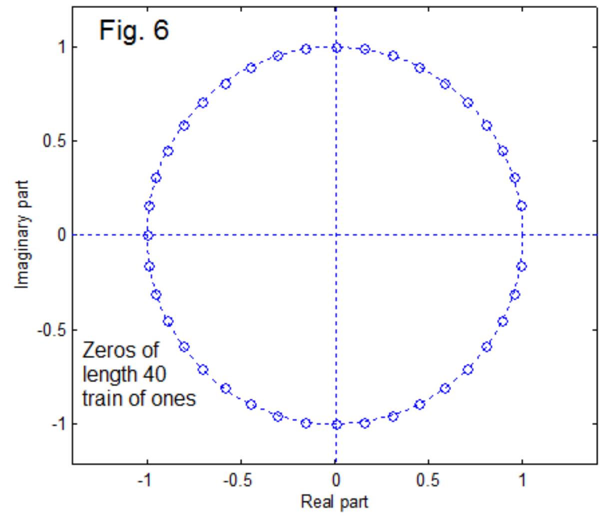


Fig. 4

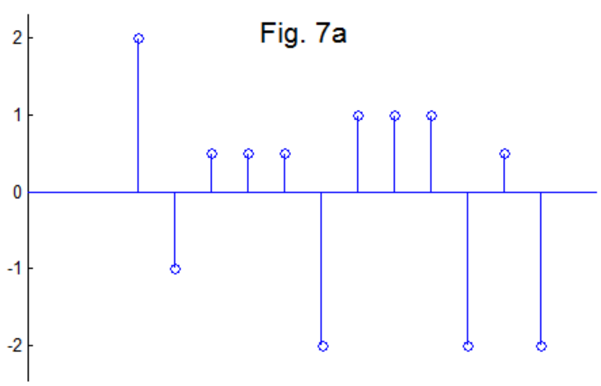
All the arguments relating to time-domain cancellations, in pairs, or overall, that we saw above apply to this reversed situation. It is just a different perspective. Note that here the input is a train of samples equal to 1, and the length is longer than that of the impulse response, so in the frequency domain, the input is of frequency zero.

The zeros of the 12 samples of the sine waves, as an impulse response as in Fig. 4, are shown in Fig. 5. Indeed it looks like a complete "ring" of zeros on the unit circle except for the frequencies of +1/12 and -1/12, as we expect. In contrast, the zeros of the input are on a ring except for z=1 (DC) as in Fig. 6.

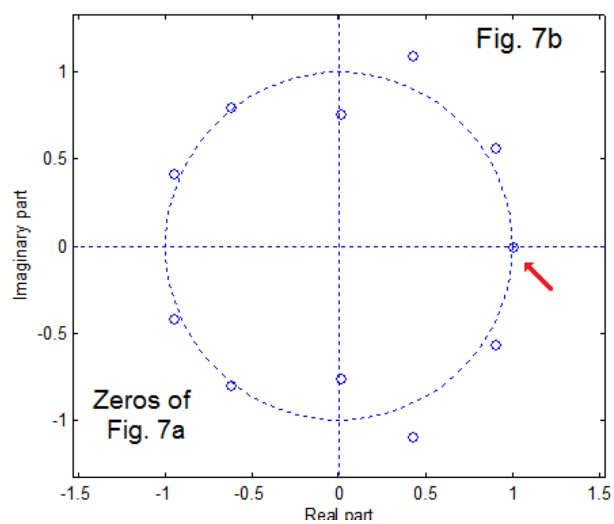




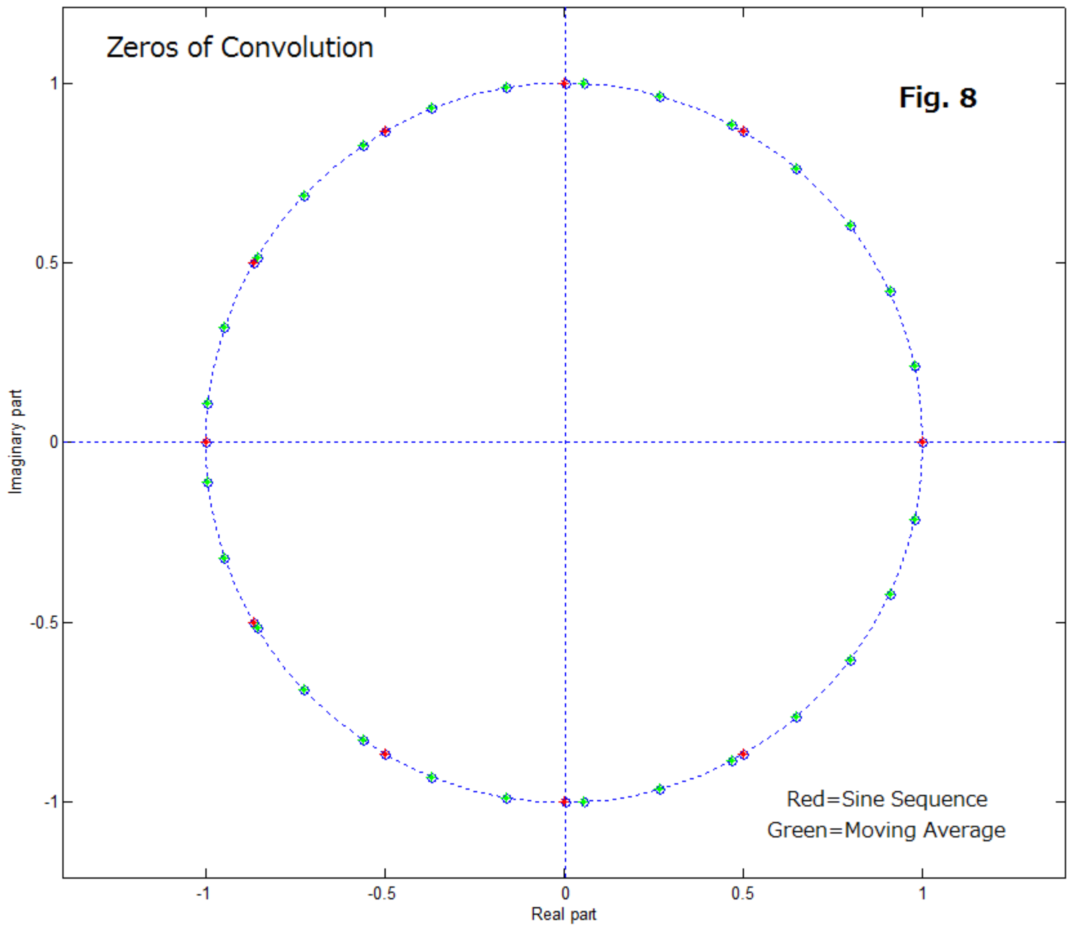
We note that the zero at z=1 in Fig. 5 would mean that DC is blocked. Are the other zeros essential to the cancellation? That this is not true can be seen from the time-domain arguments, or by just looking at another case where the impulse response values sum to zero (Fig. 7).



Arbitrary length 12 sequence summing to zero



So we note that this case has a zero at z=1.



ZEROS OF THE CONVOLUTION

Having now shows the more-or-less reciprocal relationship between the input and the filter, we are in a position to recognize that what matters here is the fact that the output in either case is the time-domain convolution of the two component sequences, and as such, the frequency domain descriptions multiply, and the zero plot of the convolved sequence includes all the zeros of both. In Fig. 8 we show the zeros resulting from the convolution of a length-12 sine sequences (as in Fig. 7a) with a length 29 moving average (similar to Fig. 6, except 28 zeros). The zeros of the moving average are shown in green, and those of the sine sequence are in red.

ZEROS OF UNBALANCED SEQUENCE

As a "loose end", we have not really shown an example of the zeros of a sequence that does not sum to zero. Instead, we claim (Fig. 5 and Fig. 7b) that balanced sequences have a zero at z=1. Fig. 9a shows an unbalanced sequence, 12 samples of a sequence of frequency 12/14. It represents two cycles short of a full cycle and the samples sum to 1.2157. The plot of the zeros (fig. 8b) shows no zero at z=1.