ELECTRONOTES
1016 Hanshaw Rd
Ithaca, NY 14850
607-266-8492

APPLICATION NOTE NO. 364

April 5, 2006

FOURIER SERIES AND DFT

INTRODUCTION

What is the relationship between the Fourier Series and the DFT? Can the Fourier Series coefficients be computed from the DFT?

Comment 1: DFT as an Aliased Fourier Series

First, note that the DFT and the Fourier Series are certainly not the same thing, a fact that is apparent from the equations below. The complex Fourier Series equations are:

$$c(k) = (1/P) \int_{-P/2}^{P/2} f(t)\, e^{-2\pi jkt/P}\, dt \qquad \text{Fourier Series coefficients (analysis)} \qquad (1a)$$

$$f(t) = \sum_{k=-\infty}^{\infty} c(k)\, e^{2\pi jkt/P} \qquad \text{Fourier Series sum (synthesis)} \qquad (1b)$$

while the DFT equations are:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j(2\pi/N)nk} \qquad \text{DFT (analysis)} \qquad (2a)$$

$$x(n) = (1/N) \sum_{k=0}^{N-1} X(k) e^{j(2\pi/N)nk} \qquad \text{Inverse DFT (synthesis)} \qquad (2b)$$

All comparisons of the various "Fourier Transform Pairs" show remarkable similarities, and also important differences. A notably similarity here is that the two synthesis equations are sums of complex exponentials weighted by coefficients that correspond to discrete harmonic frequencies. Can we relate c(k) to X(k)?

Since f(t) is a continuous-time waveform, it can be sampled at t=nT to give x(n)

$$x(n) = f(t=nT) = \sum_{m=-\infty}^{\infty} c(m)e^{2\pi jmnT/P} \tag{3a}$$

and it is useful here to consider the special case of P=NT so that we have samples of exactly one period of f(t). Thus:

$$x(n) = \sum_{m=-\infty}^{\infty} c(m)\, e^{2\pi jmn/N} \tag{3b}$$

We can take the DFT of this x(n) to give:

$$X(k) = \sum_{n=0}^{N-1} \sum_{m=-\infty}^{\infty} c(m)\, e^{2\pi jmn/N}\ e^{-j(2\pi/N)nk} \tag{3c}$$

combining exponentials and rearranging summations:

$$X(k) = \sum_{m=-\infty}^{\infty} c(m) \sum_{n=0}^{N-1} e^{j(2\pi/N)(m-k)n} \tag{3d}$$

and we know that:

$$\sum_{n=0}^{N-1} e^{j(2\pi/N)(m-k)n} = \quad N \text{ if } (m-k)= 0 \bmod N, \qquad 0 \quad \text{else} \tag{`A'}$$

We thus end up with

$$X(k) = N \sum_{m=-\infty}^{\infty} c(k+mN) \tag{3e}$$

We could perhaps have pretty much guessed at least the general form of this result based on sampling theory. We know that c(k) is the "spectrum" of the continuous waveform and sampling replicates this spectrum about all multiples of the sampling frequency. In general the Fourier Series is not bandlimited, so we expect frequency aliasing. With the sampling condition P=NT, the aliased frequencies exactly overlap the originals. The DFT samples this spectrum at the same frequencies (k/NT). So a DFT is an aliased Fourier Series.

First of all, we can see that this is not an efficient way to compute the DFT! Further, it suggests that we are not going to be able to exactly compute the Fourier Series coefficients from the DFT, due to the aliasing. About all we can argue is that since the c(k) generally fall off as 1/k or as $1/k^2$, and because N may be quite large, it may be true that:

$$c(k) \approx (1/N)\, X(k) \tag{4}$$

and this may be good for at least the smaller values of k. We make N large [take a lot of samples for one period of f(t)] and don't trust the results except for k small relative to N. The method is definitely going to fail if k approaches N, no matter how large N is. However, the first few values might be all we need to know. Perhaps we are just checking our hand calculations.

Comment 2: Fourier Series of a Piecewise Constant Waveform

It should be no surprise that the N samples x(n) from which the DFT X(k) is obtained do not contain enough information to calculate the c(k). This is first because we can imagine an infinite number of functions f(t) that pass through x(n)=f(nT), and the integral used for c(k) accommodates these different functions, while all have the same samples x(n). Secondly, how could an infinite number of c(k) ever be obtained from just N values of X(k)?

However, in one case, of some practical interest, we can get the c(k) exactly. This is the case where f(t) between the samples is held at the value of the samples. This is what we call a "hold" operation, and it is generally what we expect when we output a sequence of numbers to a D/A converter, and in similar cases. Indeed, many engineers think of a sequence x(n) as "really" being a sequence of steps. In this case, there is no additional information that is "hidden" between the samples, and we should have everything we need. In addition, this stepped approximation gets around the problem of the infinite number of c(k) through the periodicity of X(k), as we shall see.

Since we want to use the DFT, indexed on 0 to N-1, it is convenient to calculate Fourier Series coefficients on the interval 0 to P rather than from -P/2 to P/2.

$$c(k) = (1/P) \int_0^P f(t)\, e^{-2\pi jkt/P}\, dt \tag{5a}$$

Now f(t) is a constant x(n) for time t=nT to t=(n+1)T so the integral is broken into a sum of smaller integrals:

$$c(k) = (1/NT) \sum_{n=0}^{N-1} x(n) \int_{nT}^{(n+1)T} e^{-jk(2\pi/NT)t} \, dt \qquad (5b)$$

The sub-integrals of exponentials are evaluated at their endpoints giving us:

$$c(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j(2\pi/N)nk} \left(1 - e^{-jk(2\pi/N)}\right) / (2\pi jk)$$

$$= X(k)\, F(k) \qquad (5c)$$

where X(k) is exactly the DFT of x(n), and F(k) is a spectral shaping function:

$$F(k) = \left[\left(1 - e^{-jk(2\pi/N)}\right) / (2\pi jk) \right]$$

$$= (1/N)\, e^{-jk\pi/N} \left[\sin(\pi k/N) / (\pi k/N) \right] \qquad (6)$$

There is a lot here. We note that c(k) was needed for all k, so X(k) is also needed for all k, not just for the usual k=0 to N-1. Of course, X(k) is actually periodic for all k, so we find that c(k) is the periodic repetition of X(k), but now shaped by the non-periodic function F(k). We also recognize the shaping function F(k) to be basically a familiar sinc function, the roll-off of a hold of duration T.

One easy test of these equations would be to choose a simple piecewise constant function and see if it gives the known Fourier Series coefficients and if the series based on these coefficients tends to the original function. For example, a continuous time square wave could be represented by a piecewise constant function of length-two with x(0)=1 and x(1)=-1. Alternatively, we should be able to use a length-four sequence with x(0)=1, x(1)=1, x(2)=-1, x(3)=-1, and also get the same answer. This can be verified by running the program code, sqtest.m given here.

PROGRAM TO TEST FOURIER SERIES

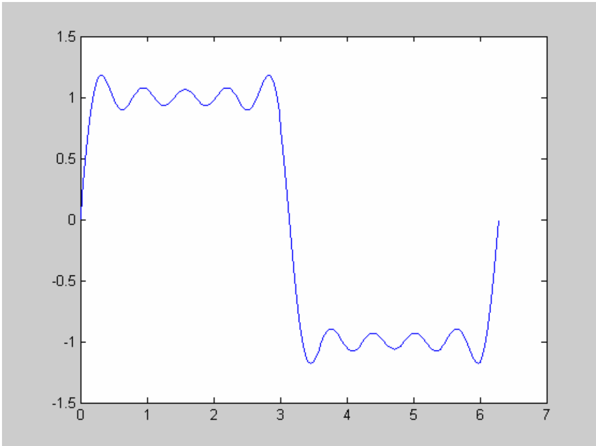
```
%sqtest.m
%  Test Fourier Series of Piecewise Constant
%     as Obtained from the DFT

% Length 2
x=[1 -1]
X=fft(x)
X=[X X X X X X X X X X]    % repeat for 19 coeffs
X=X(2:20)             % shift for c(0) at X(0)
k=-9:9
Fk=(1/2)*exp(-j*pi*k/2).*sinc(k/2)
ck=X.*Fk
% Sum for 19 coefficients
t=0:pi/100:2*pi;
f=zeros(1,201);
for k=-9:9
  f=f+ck(k+10).*exp(2*pi*j*k*t/(2*pi));
end
figure(1)
plot(t,f)



% Length 4
x=[1 1 -1 -1]     % other length 4 may be tried
X=fft(x)
X=[X X X X X X]
X=X(4:22)         % shift for c(0) at X(0)
k=-9:9
Fk=(1/4)*exp(-j*pi*k/4).*sinc(k/4)
ck=X.*Fk

t=0:pi/100:2*pi;
f=zeros(1,201);
for k=-9:9
  f=f+ck(k+10).*exp(2*pi*j*k*t/(2*pi));
end
figure(2)
plot(t,f)
```



Using sqtest.m, this is what figure(1) or figure(2) looks like

Comment 3: A Stepped Sinewave

While it is a bit difficult to imagine that DSP engineers makes frequent use of Fourier Series, and most such series that are needed are tabulated, the case of a stepped approximation to a sine wave is an example of one that is not tabulated. We would need to know this Fourier Series if we wanted to know the harmonic distortion in a stepped sine wave.

We are quite familiar with the idea of a stepped approximation to a sine wave. For example, a look-up table can be read and fed to a D/A converter (an inherent hold) to form a digital sinewave generator with analog output. In such a case, intuitively we know that it is best to have a large number of samples per cycle. But just how much distortion do we expect for a given N?

We know that the length-N DFT of a sinusoidal sequence of exactly N samples per cycle will have non-zero values only for k=1 and k=N-1. Further, we have seen that in order to get the Fourier Series coefficients, we have only to repeat this DFT and shape the amplitudes with the sample-and-hold sinc function.

$$H(k) = [\ \sin(\pi k/N) / (\pi k/N)\ ] \tag{7}$$

We notice that there is spectral energy only for harmonics k=1 (the fundamental - the sinewave we want) and for the values of k of N-1, N+1, 2N-1, 2N+1, 3N-1, 3N+1, and so on. All of these have exactly the same magnitude for the $\sin(\pi k/N)$ contribution to the shaping function H(k). As a result, the harmonics roll off as 1/k.

We can compute the Total Harmonic Distortion (THD) as:

$$THD = [A_2^2 + A_3^2 + A_4^2 + ...]^{1/2} / A_1 \quad (8)$$

where the A_k are the amplitudes of the harmonics k. Because this is a ratio, all constant scaling factors, including $\sin(\pi k/N)$ cancel top and bottom, and we end up with:

$$THD = [1/(N-1)^2 + 1/(N+1)^2 + 1/(2N-1)^2 + 1/(2N+1)^2 + ...]^{1/2} \quad (9)$$

and for large N, we can approximate:

$$1/(mN-1)^2 + 1/(mN+1)^2 \text{ approx } 2/(mN)^2 \quad (10)$$

so the THD is approximately:

$$THD \approx [(2/N^2) (1/1^2 + 1/2^2 + 1/3^2 +.......)]^{1/2} \quad (11)$$

and since the series $1 + 1/4 + 1/9 + 1/16 +..... = \pi^2/6$,

$$THD \approx \pi / (\sqrt{3}\, N) \quad (12)$$

Here we see some useful results. We first note that the THD only divides by 2 if the number of samples per cycle doubles; not a great improvement. But we also see that the first harmonic above the fundamental is at N-1, and it is perhaps surprising that there is so much open spectrum here. And, doubling N moves the harmonic about twice as far away. This is a significant improvement in that it's that much easier to filter out the harmonics even with very weak analog filtering.