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APPLICATION NOTE NO. 363

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CALCULATING THE BIT-SAVING ACHIEVED WITH NOISE-SHAPING

The usual (first-order) “noise shaping” filter [1,2] is of the form:

$$H(z) = 1 - z^{-1} \tag{1}$$

which is a simple high-pass with a zero at z=1. In the case of an k^{th} order noise shaper, we have $H(z)^k$. For practical purposes, only k=1 and k=2 (as achieved with the noise-shaping structure) are likely to be sufficiently stable to be useful.

The magnitude of the frequency response of H(z) is just:

$$|H(z)| = [(1-e^{-j\omega})(1-e^{j\omega})]^{1/2} = [2 - 2\cos(\omega)]^{1/2} \tag{2}$$

Because we will be interested in how a noise shaping filter will modify the power in the spectrum, we find it convenient to work with the squared magnitude of H(z):

$$|H(z)|^2 = [(1-e^{-j\omega})(1-e^{j\omega})] = [2 - 2\cos(\omega)] = 4 \sin^2(\omega/2) \tag{3}$$

which for a k^{th} order noise shaper is:

$$|H(z)|^{2k} = [(1-e^{-j\omega})(1-e^{j\omega})]^k = [2 - 2\cos(\omega)]^k = 4^k\sin^{2k}(\omega/2) \tag{4}$$

Fig. 1 shows the relevant functions. Now, we are interested in finding the area under this curve from ω=0 to some frequency that is π divided by the oversampling factor. If we have m octaves of oversampling, this upper frequency is $\pi/2^m$.

Here we will find it convenient to approximate the sine by its argument because we are interested mainly in fairly large oversampling factors. Thus we want the integral:

$$p = \int_0^{\pi/2^m} 4^k\sin^{2k}(\omega/2)\, d\omega \approx \int_0^{\pi/2^m} 4^k (\omega/2)^{2k}\, d\omega = (\pi/2^m)^{(2k+1)} / (2k+1) \tag{5}$$

This we want to compare to the non-oversampleing, no-noise-shaping case.

$$p_0 = \int_0^{\pi} 1\, d\omega = \pi \tag{6}$$

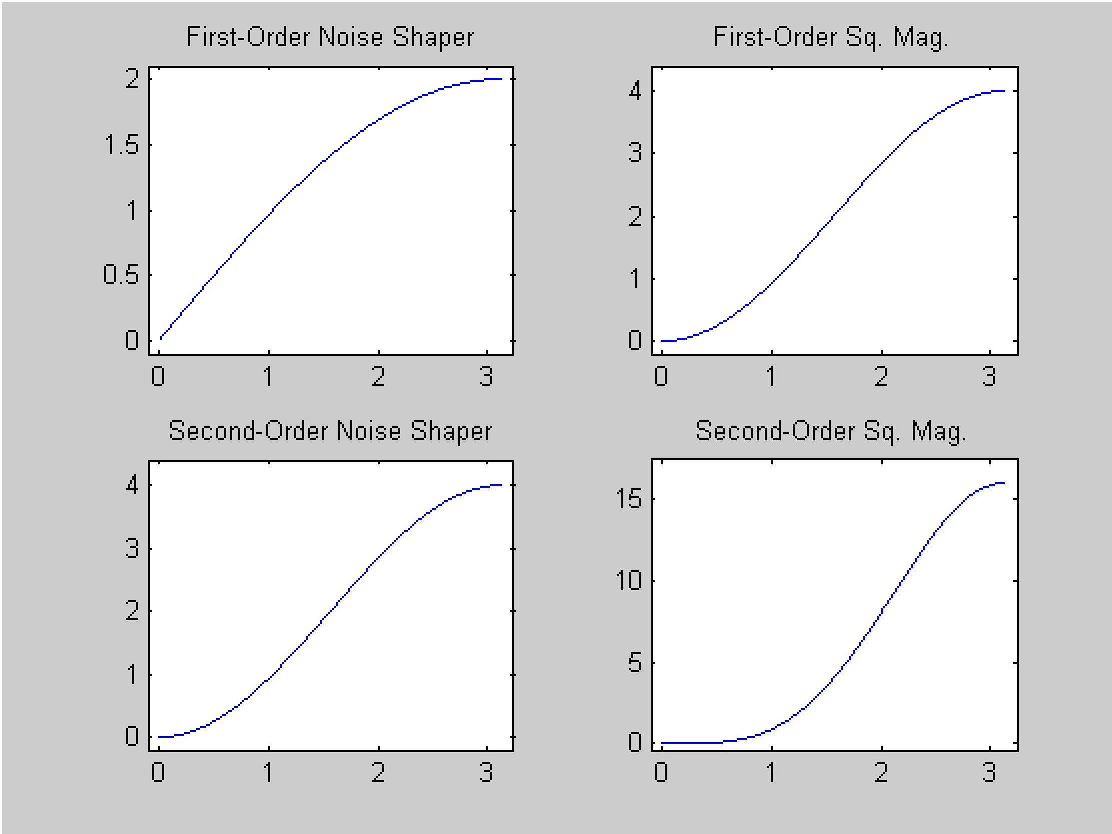

Fig. 1   First- and Second-Order Noise Shaping Curves

so we have p/p_0 as:

$$p/p_0 = \pi^{2k} / [(2^{m(2k+1)})(2k+1)] \quad (7)$$

Now, this reduction of noise power could also have been achieved if we had added ΔB bits of resolution to the signal, which would reduce the noise power by $2^{-2\Delta B}$, so

$$2^{-2\Delta B} = p/p_0 = \pi^{2k} / [(2^{m(2k+1)})(2k+1)] \quad (8)$$

Taking the log base 2 of both sides:

$$-2\Delta B = \mathrm{Log}_2[\pi^{2k}/(2k+1)] - \mathrm{Log}_2[2^{m(2k+1)}] \quad (9)$$

or:

$$\Delta B = (1/2)\,\mathrm{Log}_2[2^{m(2k+1)}] - (1/2)\,\mathrm{Log}_2[\pi^{2k}/(2k+1)]$$

$$= m(k+1/2) - (1/2)\,\mathrm{Log}_2[\pi^{2k}/(2k+1)] \quad (10)$$

This is the details of a result quoted in Orfanidis [1] and given in part in a previous Application Note No. 345 [2].

{Note: Please check any copies of AN-345 you may have. We have come across copies that have $\text{Log}_2 m$ instead of m in equation (9) of that AN, equivalent to equation (10) here. We are defining m as “octaves.” If it were the oversampling factor here, then the Log_2 would be correct. While our originals and the copies we are currently shipping are correct, it is possible that a few copies were made from an earlier incorrect version, for which we apologize.}

We mentioned that Orfanidis used a reasonable approximation of $\sin(x) \approx x$ for small x. Because we really are interested in only k=1 and k=2, we can do the exact problem using tabulated integrals [3].

$$p = \int_0^{\pi/2^m} 4^k \sin^{2k}(\omega/2)d\omega \qquad (11)$$

For which the k=1 case gives us:

$$p_1 = \int_0^{\pi/2^m} 4 \sin^2(\omega/2)d\omega \quad = 2 [\pi/2^m - \sin(\pi/2^m)] \qquad (12)$$

while the k=2 case gives us:

$$p_2 = \int_0^{\pi/2^m} 4^2 \sin^4(\omega/2)d\omega = 6\pi/2^m - 8\sin(\pi/2^m) + \sin(2\pi/2^m) \qquad (13)$$

Accordingly we have only to plug in the value of m to $p = p_1$ or p_2, and calculate the added bits as :

$$\Delta B = -(1/2) \log_2(p/\pi) \qquad (14)$$

Table 1 shows the results of calculations using the exact formulas, equations (12) and (13), the Orfanidis formula, equation (10) and the “Hauser Rule of Thumb” [4]. [This rule is that first-order gives 1.5 bits/octave with a one-bit penalty, while second order gives us 2.5 bits/octave with a 2 bit penalty.] Since most practical cases will involve at least a factor of 16 of oversampling, we see excellent agreement regardless of the formulas used.

TABLE 1

Bit Savings for First- and Second-Order Noise Shaping
for m Octaves of Oversampling

Octaves of Oversampling ↓	First-Order Noise Shaping			Second-Order Noise Shaping		
↓	Exact	Orfanidis	Hauser	Exact	Orfanidis	Hauser
m=0	-0.5000	-0.8590	-1.0000	-2.1420	0	-2.0000
m=1	0.7302	0.6410	0.5000	0.5704	0.5704	0.5000
m=2	2.1632	2.1410	2.0000	2.9110	2.9110	3.0000
m=3	3.6465	3.6410	3.5000	5.3712	5.3712	5.5000
m=4	5.1424	5.1410	5.0000	7.8613	7.8613	8.0000
m=5	6.6413	6.6410	6.5000	10.3588	10.3588	10.5000
m=6	8.1411	8.1410	8.0000	12.8582	12.8582	13.0000
m=7	9.6410	9.6410	9.5000	15.3580	15.3580	15.5000
m=8	11.1410	11.1410	11.0000	17.8580	17.8580	18.0000
m=9	12.6410	12.6410	12.5000	20.3580	20.3580	20.5000
m=10	14.1410	14.1410	14.0000	22.8580	22.8580	23.0000

REFERENCES

[1] B. Hutchins, "Improved Signal/Noise Ratio with First-Order Noise Shaping: An Example," Electronotes Application Note No. 345, October 1997

[2] S. Orfanidis, Introduction to Signal Processing, Prentice-Hall (1996) pp 67-73

[3] CRC Standard Mathematical Tables, 25th Ed., CRC Press (1978)

[4] M. Hauser, "Principles of Oversampling A/D Conversion," J. Audio Eng. Soc., Vol. 39, No. 1/2, Jan/Feb 1991, pp 3-26.