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APPLICATION NOTE NO. 361

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PASSIVE SENSITIVITIES AND TUNING EQUATIONS

INTRODUCTION

The notion of "classical sensitivity" has been around a very long time as a general means of selecting among various circuit configurations, with regard to how well they meet specifications while using imprecise components (components with "tolerances") [1]. Not only do these calculations provide some ideas relating to the suitability of a particular network configuration, but they can also provide "tuning equations" for calculating small correcting components for "trimming up" individual units [2].

A SIMPLE VOLTAGE DIVIDER EXAMPLE

To begin, let's consider the humble voltage divider (Fig. 1). Here the output voltage is determined from the input as:

$$V_{out} = \alpha V_{in} = V_{in} R_2 / (R_1 + R_2) \qquad (1)$$

or:

$$\alpha = R_2 / (R_1 + R_2) \qquad (2)$$

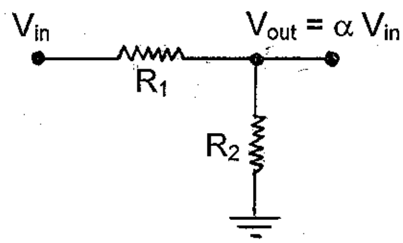

Fig. 1 Voltage Divider

Classical sensitivity is defined as a ratio of fractional changes (a "slope") as:

$$S_x^y = (\Delta y/y) / (\Delta x/x) \qquad (3a)$$

where we are asking how some performance parameter y varies with changes in some component value x. More exactly, we can look at it as saying that a certain percentage change in y, (100 Δy/y) is expected for a certain percentage change in x (100 Δx/x) at a certain point (the 100's of course cancel). This is usually calculated as:

$$S_x^y = (x/y) (\partial y / \partial x) \qquad (3b)$$

Applying this to α we have:

$$S^{\alpha}_{R1} = (R_1/\alpha)\,(\partial\alpha / \partial R_1) = -R_1/(R_1+R_2) = \alpha - 1 \tag{4a}$$

$$S^{\alpha}_{R2} = (R_2/\alpha)\,(\partial\alpha / \partial R_2) = R_1/(R_1+R_2) = 1 - \alpha \tag{4b}$$

These calculations are not atypical even though it is a particularly simple case. Note that we plugged the nominal value for α into the calculation of (4a) and (4b), and that we then wrote the result in terms of the nominal value of α. Note that the two sensitivities are negatives of each other in this case, and that since α must be between 0 and 1, the sensitivities range in magnitude from 0 to 1. We might have expected the largest sensitivity (magnitude) to be associated with the larger of the two resistors for a particular α, but this is not the case. The sensitivities are of the same magnitude, and are minimal (and of magnitude 1/2) when $\alpha = 1/2$.

We can understand this by noting some limiting cases. Suppose α is very small, so that R_2 is much less than R_1. Now if R_2 is doubled, it is still very small compared to R_1, and the series current is only very slightly smaller, so α doubles. This is what equation (4b) says for small α – the sensitivity is almost up to 1, and α essentially changes by the same percentage that the resistor R_2 varies. Further, if R_1 were to double, and current would go to half its original value, and α would become half its original value, again changing by the same percentage as the resistor R_1. Thus we understand the sensitivities being very close to 1 in magnitude. Similar arguments hold for α close to 1. What happens when α is close to 1/2? This is covered in the example below.

Normally we would not think of a voltage divider as a filter, or even a system, but the value of α is certainly a "performance parameter" of the divider. Note that from the sensitivity calculations, we can obtain a tuning equation. For example, suppose we do want α=1/2 but experimentally, we find α=0.48 (low by 4%). (This is not a hard problem of course, so we just use it for an initial illustration.) Intuitively we recognize that we would need to add a small series resistor to R_2 for this case. But how much – 4% of the nominal value of R_2? No, the sensitivity of α to R_2 is 1 - α which is 1 - 1/2 or just 1/2 for this case (see equation 4b). Thus a series resistor of 8% of R_2 should be added.

Let's do an actual example. Suppose R_1 and R_2 are nominally 10k, but the actual values (within an expected 5% tolerance) are R_1=10.41k and R_2=9.61k. This would give us α=0.48 in reality. Note that we do not know the true values of these resistors – we just know the measured performance parameter is α=0.48. According to our theory, we need to add 8% of (the nominal value of) 10k or 800 ohms to R_2. If we did this, we would get

$$\alpha = (9610+800) / [10410 + (9610+800)] = 0.5 \qquad (5a)$$

which is exactly what we wanted to get. In reality, we would choose a 5% resistor close to 800 ohms, which would be a nominal (standard value) 820 ohms. Let's suppose this is only 790 ohms in reality.

$$\alpha = (9610+790) / [10410 + (9610+790)] = 0.4998 \qquad (5b)$$

which is still an excellent correction of course. Thus we see the potential of combining a measurement of a performance parameter, with a tuning equation (sensitivity calculation) to obtain a "trimming" component, with a performance result far better than the tolerances of any of the components involved.

We should perhaps do one more example with a different α. Suppose we want α = 0.25, so we choose R_1=30k and R_2=10k nominally. But, if R_1=29.7k and R_2=10.3k in the actual components, we would get α=0.2575, which is 3% high. In contrast to the first example, here the gain is high and we need to add a series resistor to R_1. But also here, we do not change by 6% (that is, we do not double the percent error as we did for α=0.5). Instead we note from equation (4a) that the magnitude of the sensitivity is $1-\alpha$ = 1-1/4 = 0.75. Thus we need to calculate the trimming resistor as 3%/0.75 = 4% of 30k or 1200 ohms. Indeed this gives α=0.25. The reader is invited to consider 5% variations from 1200 ohms, and indeed, to do more examples.

UNDERSTANDING THE TUNING EQUATIONS AND THEIR APPLICATIONS IN PRACTICE

By simply rearranging equation (3a) we see that our so-called tuning procedure is summarized by the "tuning equation"

$$\Delta x = x (\Delta y/y) / S^y_x \qquad (6)$$

where S^y_x is calculated using equation (3b). That is, we get Δx, the amount to change x, in order to get the desired Δy. We may need to use some common sense to interpret Δx. In our examples, we have made Δx a change is resistance that is positive, and we saw this as a series trimming resistor. If Δx is negative, we would understand that for a resistor we would need to calculate a parallel resistor to reduce the total resistance. Or, we might just attack a different component (R_1 instead of R_2 in our examples above).

Another point is that when we calculate using equation (3b), we may make errors in our math. How do we check our results? Well, we can always just go back to the design equations and jiggle the component values slightly – "sensing" the derivative about the nominal point. In our example, equation (2) is the design equation. If we jiggle R_1 for example, we can see how much α changes, and see if it agrees with equation (3b). More or less, this is what we have done in our examples, just to show how things work.

Employment of tuning equations is most useful when we have many units to be trimmed. The reader may have recognized that by the time the math is worked out and verified, any single circuit could probably have been trimmed by hand – trial and error. We gain from our initial calculations when we measure, calculate, and trim successive units, each one being done with excellent results on the first try.

It is also worth remembering that with a normal scatter of component tolerances, we would expect the performance parameter to vary both high and low, likely forcing us to choose between two different trimming strategies depending on the measured parameter result. This we illustrated in our example where once α was low, and once it was high. It is probably more useful to take steps to assure that all the units are wrong in the same direction. In the case of the voltage divider, we might have chosen R_1=10k and R_2=9.1k, even though we wanted α=0.5. In this way, we would always be adding a series correction to R_2, moving α upward.

In most actual systems we would have in mind something like a filter that had such performance parameters of cutoff frequency, "Q", and gain (not just the trivial α of the voltage divider). In these cases, the tuning equation approach may be much more useful, as trial and error could be very tedious. Note that in these cases, we would likely do something like choosing the nominal cutoff frequency always slightly high, so that it could be then trimmed downward with a series resistor. However, in some cases it is probably easier to actually install parallel components. More on actual filters below.

SALLEN-KEY

Probably no active filter is more famous than the Sallen-Key Low-Pass [3,4,5]. Accordingly we can use this to illustrate some sensitivity and tuning applications. Fig. 2a shows the usual simplified circuit while Fig. 2b shows a more detailed circuit. The transfer function of Fig. 2a is:

$$T(s) = \frac{K/R^2C^2}{s^2 + (3-K)s/RC + 1/R^2C^2} \qquad (7a)$$

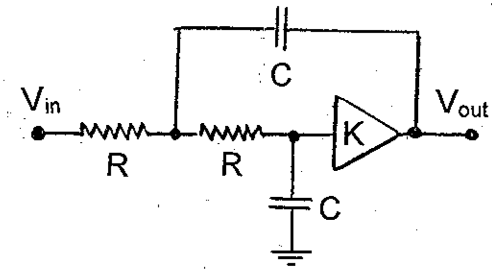

Fig. 2a Simplified Sallen-Key

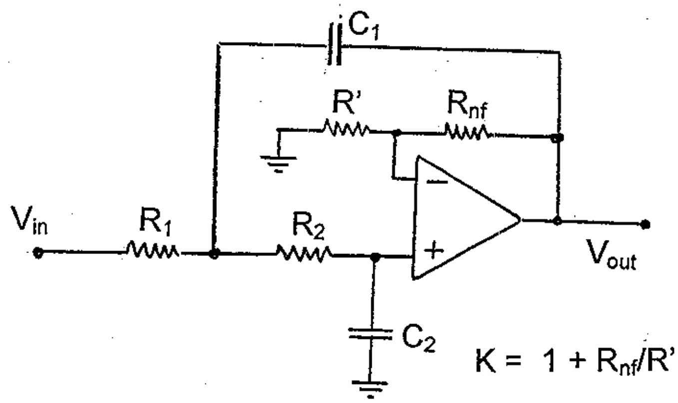

Fig. 2b Full Sallen-Key

From this we can obtain the "design equations"

$$\omega_0 = 1/RC \tag{7b}$$

$$D = 1/Q = (3 - K) \tag{7c}$$

where ω_0 is the "pole radius" and the "damping" $D = 1/Q$ relates to the angle of the poles [6]. These calculations correspond to the "equal R, equal C" selection of the components. That is, we intend that both the resistors R and both the capacitors C be the same value – a very reasonable approach. Yet as we shall see, this can mislead us. The resistors are not the same in practice simply because any two (even nominally marked the same) will have at least slightly different values. But, far more importantly, they are different because they are in different places in the network. This does matter.

Accordingly, we can look at Fig. 2b. For one thing we have shown the actual op-amp realization of the non-inverting amplifier K. In addition, the R resistors and C capacitors are separately numbered (nominally they are still the same). The transfer function is now:

$$T(s) = \frac{K/R_1R_2C_1C_2}{s^2 + s\,[(1-K)/R_2C_2 + 1/R_1C_1 + 1/R_2C_1] + 1/R_1R_2C_1C_2} \tag{8a}$$

with corresponding design equations:

$$\omega_0 = 1/\sqrt{[R_1R_2C_1C_2]} \tag{8b}$$

and

$$D = (1-K)\sqrt{[R_1C_1/R_2C_2]} + \sqrt{[R_2C_2/R_1C_1]} + \sqrt{[R_1C_2/R_2C_1]} \quad (8c)$$

where

$$K = 1 + R_{nf}/R' \quad (8d)$$

All we have done to get the design equations is to fit the actual transfer function denominators to the generic denominator:

$$d(s) = s^2 + Ds\omega_0 + \omega_0^2 \quad (9)$$

Now there are a couple of key points. First, when we set the resistors equal and the capacitors equal in equations (8), we get, as we must, the results of equations (7). Second, when we look at the simplified case (equations 7b and 7c) we suppose that the pole radius (cutoff frequency) depends only on R and C, and that the pole damping depends only on K. But equation (8c) shows us that the damping does depend on the actual matching of R and C values (though more strongly on K).

A third point relates to the fact that when we try to calculate the sensitivities, we can get fooled by the simplified case. Using equation (3b), applied to equation (7b), we get:

$$S_R^{\omega_0} = (R/\omega_0)\,\partial\,(1/RC)\,/\,\partial R = R^2C(-C)/R^2C^2 = -1 \quad (10a)$$

and <u>this is wrong</u>, or at least useless. What it says is that if both the resistors R were wrong by the exact same (relatively small) percentage, the frequency would change in the opposite direction by the same percentage. But the resistors are not exactly the same – that is the whole point. What we need is to use equation (8b):

$$S_{R_1}^{\omega_0} = (R_1/\omega_0)\,\partial\,(\,[R_1R_2C_1C_2]^{-1/2})\,/\,\partial R_1 = -1/2 \quad (10b)$$

and this is the right answer. When only one resistor changes, the frequency only changes by half the percentage. The same sensitivity calculation for ω_0 apples to R_2, C_1, and C_2 of course.

Notice that the sensitivity of ω_0 to K is actually zero. This is not exactly saying that the cutoff frequency of the low-pass does not vary with K, because as K varies, the "characteristic" of the low-pass varies (e.g., it may peak more) and a cutoff defined in a particular way (relative to the pole radius that does not change with K) may vary at least slightly.

Certainly however the damping is sensitive to K. In particular, using equation (7c) we have:

$$S_K^D = (K/D)\,\partial(3 - K)/\partial K = -K/(3\text{-}K) \qquad (11)$$

which may be a recipe for disaster. In particular, in many higher-order filters that are formed by cascading second-order sections as in Fig. 2, the highest Q (lowest D) section may well have K uncomfortably close to 3. Even with K just 2.9, note that the sensitivity is -29. A tiny change in K can have a major change in D. So this is revealing, but is perhaps something we already knew: we don't want that center term of the denominator to become zero or to go negative.

It could well be argued that it is not the sensitivity relative to K that we are concerned with, but rather the sensitivity relative to R_{nf} and R'. Looking at this issue will permit us to try a few additional ideas. First, it is easy to show that there is a "chain rule" that applies to sensitivities so that, for example:

$$S_{R'}^D = S_K^D \; S_{R'}^K \qquad (12)$$

and it is also easy to show that:

$$S_{R'}^{1/\alpha} = -S_{R'}^{\alpha} \qquad (13)$$

These relationships are familiar from differential calculus. The significance of these with regard to the present problem is that the K of our Sallen-Key is just the $1/\alpha$ of the voltage divider studied originally [and $\alpha = R'/(R'+R_{nf})$]. Thus using equation (4b) along with (11), (12) and (13) we arrive at:

$$S_{R'}^D = S_K^D\,(-S_{R'}^{\alpha}) \quad = [-K/(3\text{-}K)]\,[-(K\text{-}1)/K] = [K\text{-}1]/[3\text{-}K] \qquad (14)$$

We see that the magnitude of this last sensitivity lags slightly the magnitude of the sensitivity to K, as it should. Perhaps this is most easily seen since the 1 in equation (8d), all of K, does not vary with R_{nf}/R', part of K. (The result of equation (14) can also be calculated directly of course.)

We can try this with an example. Suppose we want D=0.1 (a Q of 10, fairly high). This means we want K=2.9. We could choose R_{nf}=19k and R'=10k. But let's suppose R' is high by 0.1% (just 10 ohms!) and we actually have 10.01k. Now K=2.8981 (slightly low) so D=0.1019, which is 1.9% high. Thus we have a fairly sensitive result, a change in R' results in 19 times that change in D, by calculation from the design equations. What does the tuning equation (14) say? Plugging 2.9 into equation (14) we do get S^D_R=19, in agreement with the example.

THE DELIYANNIS BANDPASS

Our final example here is a popular bandpass filter known as the "Deliyannis" bandpass which has proven useful in single units [7,8] as well as in filter banks [9]. In particular, when used in a bank of dozens of filters, we really do need some sort of tuning equation and trim procedure. Trial and error in such a case would clearly be too tedious.

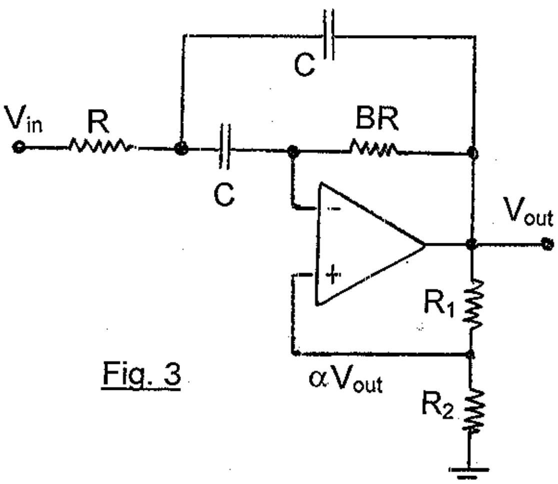

Fig. 3

The Deliyannis bandpass is shown in Fig. 3. It is basically a multiple-feedback-infinite-gain bandpass with positive feedback. The transfer function is:

$$T(s) = \frac{-s/[(1-\alpha)RC]}{s^2 + \dfrac{[2(1-\alpha) - \alpha B]s}{(1-\alpha)BRC} + 1/BR^2C^2} \tag{15}$$

From this we easily derive the design equations:

$$\omega_0^2 = 1/BR^2C^2 \tag{16}$$

$$Q = (1-\alpha)\sqrt{B} \;/\, [\,2 - \alpha(B+2)] \tag{17}$$

$$g = B\,/\,[2 - \alpha(B+2)\,] \tag{18}$$

where g is the passband gain (gain at center frequency).

One of the design goals of constructing a filter bank is usually to have the gains of all filters the same, and this is probably more important than getting all the Q's the same. Further, the Q is generally hard to measure, while the gain is easily measured. Accordingly it is often the case that we do trimming based on the gain. Notice that Q and g have a similar dependence on α [$(1-\alpha)$ is close to 1], so we expect that trimming g will also move Q in the right direction as well. Typically we would build the filter with R_2 intentionally set one drawer low (the 5% value below the closest 5% value). Then we measure the gain, which should be low. Based on the additional gain needed, we calculate and install a series correction resistor.

Thus what we need is the sensitivity of g to R_2. While our first "trivial" example was just a voltage divider, we did manage to use the voltage divider result in the Sallen-Key filter, and we will use it again here. We start out to find the sensitivity of g to α Using equation (18)

$$S_{\alpha}^{g} = (\alpha/g)\,\partial g/\partial\alpha = \alpha(B+2) / [2 - \alpha(B+2)] \qquad (19a)$$

Equations (4b) and (12) then lead to:

$$S_{R_2}^{g} = S_{\alpha}^{g}\, S_{R_2}^{\alpha} = \alpha(B+2)(1-\alpha) / [2 - \alpha(B+2)] \qquad (19b)$$

where α is $R_2/(R_1+R_2)$. Thus, if we knew that we needed a certain delta g to be added, we could calculate the additional series resistor ΔR_2 needed as:

$$\Delta R_2 = R_2\,(\Delta g/g) / S_{R_2}^{g} \qquad (20)$$

Note that the nominal values of g and of R_2 are used to calculate this correction. Note also that if we needed to trim the center frequency, we would do that first, since this will almost certainly involve changing R or BR, and hence B. But then B is fixed as we adjust the gain.

Calculating sensitivities with partial derivatives can be tedious [1], and we have to keep in mind what we are trying to accomplish. Thus we may at times become tired and arrive at an answer that requires some verification. This is not hard to do. For the example above, we calculate a first value of g using equation (18), with $\alpha = R_2/(R_1+R_2)$ of course. Then we change R_2 slightly. How slightly? Well, as little as a tenth of a percent, and possibly as little as one part per million, or less! You then calculate a second value of g for the new value of R_2. You now have g, R_2, Δg, and ΔR_2, and can calculate the sensitivity at that point. Does it agree with the calculation using partial derivatives? It should. A typical program (in Matlab) is appended to this note.

REFERENCES

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[6] B. Hutchins, "Analog Signal Processing, Chapter 3, Transfer Functions for Standard Filters," Electronotes, Vol. 19, No. 192, Feb. 2000.

[7] "High-Q Bandpass Filter," Electronotes Application Note No. 37, May 4, 1977

[8] "Analysis of the "Deliyannis" Filter," Electronotes Application Note. No. 145, Sept 4, 1979.

[9] B. Hutchins, "A 39-Channel Variable-Q Filter Bank (Part 1)," Electronotes, Vol. 12, No. 115, July 1980, pp 3-10

APPENDIX

The program below was used to test the validity of equation (19). The original test point is a 300 Hz bandpass with Q of 100, obtained as: deli(10600,.005E-6,100,43000,817)

```
function deli(R,C,B,R1,R2)
% Nominal Design
a=R2/(R1+R2)
f0=1/(2*pi*sqrt(B)*R*C)
Q=((1-a)*sqrt(B)) / ( 2*(1-a)-a*B )
g=B/( 2*(1-a) - a*B )

% "Jiggle" R2
R21=R2*.9999999
a1=R21/(R1+R21)
f01=1/(2*pi*sqrt(B)*R*C)
Q1=((1-a1)*sqrt(B)) / ( 2*(1-a1)-a1*B )
g1=B/( 2*(1-a1) - a1*B )

% Sensitivity based on Example
SGR2E=( (g-g1)/g )/ ((R2-R21)/R2 )

% Sensitivity Based on Calculation
SGR2C=(a*(1-a)*(B+2)) / (2 - a*(B+2))
```