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DFT LENGTH CONVERSION

Recently released hand-held calculators may offer a FFT function, but often only for lengths that are powers of 2. When lengths that are not powers of 2 are required, it is sometimes suggested that one need only "pad" the original sequence by appending zeros up to an available length. There is some validity to this suggestion, in that if we are only looking for a dense sampling of the DTFT, we will get this. However, it is not true that the correct DFT is obtained directly, nor is it the case that a simple procedure yields the correct DFT. The method below gives the correct DFT, but it would likely be impractical on a calculator.

Suppose It is desired to obtain the 7-point DFT, X(k), of a 7-point time sequence x(n) by using the 8-point DFT, Y(k), of an 8-point time sequence y(n) where:

$$y(n) = x(n) \qquad n=0,1,\ldots 6 \tag{1a}$$
$$y(7) = 0 \tag{1b}$$

Thus y(n) is simply x(n) with a zero added at the end. That it should be possible to do this is suggested by the fact that both x(n) and y(n) have the <u>exact same DTFT</u>, the DTFT being defined as:

$$W(e^{j\omega}) = \sum_{n=-\infty}^{\infty} w(n)\, e^{-jn\omega} \tag{2}$$

That is, w(n) yields $W(e^{j\omega})$ for <u>all ω</u>, so we certainly can obtain 7-point and 8-point samplings, the DFT's of interest, from $W(e^{j\omega})$.

It is useful to employ an example, which will also serve as a counter-example to any notion that X(k) might simply be Y(k) for k=0,1,....6. Suppose x(n) = [1 3 -2 1 -1 3 -3] so that y(n) = [1 3 -2 1 -1 3 -3 0]. In this case we find X(k) and Y(k) as:

k →	0	1	2	3	4	5	6	7
X(k) →	2 00	0.778-0.684j	0 099-6 46j	1 62-8.46j	1 62+8 46j	0.099+6 46j	0.778 0 684j	----
Y(k)→	2 00	1.293-1 71j	5 00 -5 00j	2.71+0 293j	-12.00	2.71-0.293j	5.00+5 00j	1 293+.71j

With the exception of k=0 (the sum of the time sequence) these are different.

In general we might hope to obtain a matrix A that is 7 rows by 8 columns such that:

$$X = AY \tag{3a}$$

where we might expect the elements of the matrix A to be complex exponential factors. This is difficult to calculate, so instead, we will approach the problem through the time domain by using first a 7x8 matrix B such that:

$$x = By \tag{3b}$$

where B is a "time limiting" matrix:

$$B = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \end{bmatrix} \tag{4}$$

We note as well that $X = D_7x$ and $Y = D_8y$ where D_7 and D_8 are forward DFT matrices.

Applying these matrices to equation (3b) we have:

$$D_7^{-1} X = B\, D_8^{-1}\, Y \tag{5}$$

Or:

$$X = D_7\, B\, D_8^{-1}\, Y \tag{6}$$

So that using equation (3a) we have:

$$A = D_7\, B\, D_8^{-1} \tag{7}$$

In our example case we find A as:

$$\begin{bmatrix} 0.875 & -0.088+0.088j & 0.125j & 0.088+0.088j & 0.125 & 0.088-0.088j & -0.125j & -0.088-0.088j \\ 0 & 0.805-0.282j & -0.116+0.241j & 0.057+0.164j & 0.125+0.060j & 0.109-0.038j & 0.041-0.084j & -0.021-0.061j \\ 0 & 0.098+0.011j & 0.617-0.492j & -0.047+0.415j & 0.125+0.157j & 0.131+0.015j & 0.070-0.056j & 0.006-0.050j \\ 0 & 0.050+0.031j & 0.162-0.037j & 0.367-0.584j & 0.125+0.548j & 0.169+0.106j & 0.102-0.023j & 0.026-0.041j \\ 0 & 0.026+0.041j & 0.102+0.023j & 0.169-0.106j & 0.125-0.548j & 0.367+0.584j & 0.162+0.037j & 0.050-0.031j \\ 0 & 0.006+0.050j & 0.070+0.056j & 0.131-0.015j & 0.125-0.157j & -0.047-0.415j & 0.617+0.492j & 0.098-0.011j \\ 0 & -0.021+0.061j & 0.041+0.084j & 0.109+0.038j & 0.125-0.060j & 0.057-0.164j & -0.116-0.241j & 0.805+0.282j \end{bmatrix}$$

This is essentially the solution to our problem. We note immediately that the matrix multiply of equation (3a) is going to be more complicated than just doing a DFT.

An example program using Matlab™ shows the procedure and the example:

```
n7=0:6;
n8=0:7;
D7=exp(-j*2*pi*(n7'*n7)/7);        %forward 7-point DFT
D8I=(1/8)*exp(j*2*pi*(n8'*n8)/8); %inverse 8-point DFT
b1=eye(7);
b2=[0 0 0 0 0 0 0];
B=[b1;b2]';                        %time-limiting matrix
A=D7*B*D8I;                        %DFT conversion matrix
%  test example
x=[1 3 -2 1 -1 3 -3];
y=[x 0];
X=fft(x);
Y=fft(y);
Xtry=A*Y.';          % note: Y.' is NON-conjugate transpose
E=abs(X-Xtry.')      % error
```

The code is straightforward, with the exception that some confusion may result from the fact that the "transpose" operation (necessary to implement the matrix multiply) also conjugates. Accordingly, the ordinary transpose (.') is used rather than the transpose ('). The effect of taking the inverse transform of a conjugate is seen to be a time reversal in the case of a real time function. This is because, by definition of the inverse DFT,

$$w(n) = (1/N) \sum_{k=0}^{N-1} W(k)\, e^{j(2\pi/N)nk} \qquad (8)$$

If we conjugate both sides and also plug in –n for n we have:

$$w(-n)^* = (1/N) \sum_{k=0}^{N-1} W(k)^*\, e^{j(2\pi/N)nk} \qquad (9)$$

so in the case of a real w(n), we get w(-n). This in itself is curious because at first, there seems to be no meaning to negative n, except for n=0, since we usually write n = 0,1,...N -1. However, both the DFT and the inverse DFT are periodic with period N, so w(-n)=w(N-n). For example, if w(n) is the eight point sequence [0 1 2 3 4 5 6 7], the inverse DFT of the conjugate of W(k) is the time sequence [0 7 6 5 4 3 2 1]. If w(n) were complex, for example [0 1 2 3+7j 4 5 6 7], the inverse DFT of the conjugate of W(k) would be [0 7 6 5 4 3-7j 2 1].

In conclusion, converting DFT size in the circumstances suggested is possible and practical (although probably unnecessary since arbitrary DFT length would likely be available) with a computer-based math program. In the case of the hand-held calculator, the conversion would likely be much more trouble than simply computing the DFT directly without the FFT. As such, perhaps the principle use of these ideas relates to the understanding of the DFT that these exercises offer.

REFERENCE: B. Hutchins, "Time-Domain Zero-Padding and the DFT, " Electronotes, Vol. 19, No. 189, August 1997, pp 12-19