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LINEAR PHASE - THE TERM TAKEN OUTSIDE

In this note we remind the reader of a standard manipulative procedure for simplifying certain DTFT's (Discrete Time Fourier Transforms) when computing a frequency response from an impulse response. This is the technique of bringing out a single term representing a linear phase, and usually, simplifying what remains with one of the Euler relationships. This is so easy - and so easy to mess up.

We are first interested in a solution to a general problem of the form:

$$x^{n_1\gamma} \pm x^{n_2\gamma} = x^{\alpha\gamma}(x^{\beta\gamma} \pm x^{-\beta\gamma}) \qquad (1)$$

where n_1 and n_2 (usually integers) are known, as is γ, and we seek to determine α and β. This is of course a trivial problem. In fact, it is so trivial that we often resolve the situation in one automatic mental sweep without intermediate calculations. In fact, it is often the case that a particular problem is simplified - for example, $n_1=0$ making the first term a 1. In consequence, we often suppose that any similar problem is trivial and should be doable by just turning the mental crank a bit harder. In fact, the result may be error prone without at least a few marginal notes. First we note that $n_1=\alpha+\beta$ and $n_2=\alpha-\beta$ so that

$$\alpha=(n_1+n_2)/2 \qquad (2a)$$

and

$$\beta=(n_1-n_2)/2. \qquad (2b)$$

This is the solution.

For the most part, this comes up in digital signal processing in the case where we are evaluating frequency responses using the DTFT and we need to combine complex exponential terms into a phase and a sine or a cosine. That is, $x = e$, and $\gamma=j\omega$. For example, we may wish to combine:

$$e^{-7j\omega} - e^{-12j\omega} \qquad (4a)$$

which, using the above technique, becomes:

$$e^{-(19/2)j\omega}\,(e^{(5/2)j\omega} - e^{-(5/2)j\omega}) = 2e^{-(19/2)j\omega}\, j\sin[(5/2)\omega] = 2e^{-j(19\omega/2-\pi/2)}\sin[(5/2)\omega] \qquad (4b)$$

A second, simpler example would be:

$$1 + e^{-jN\omega} = e^{-j(N/2)\omega}\,(e^{j(N/2)\omega} + e^{-j(N/2)\omega}) = 2\,e^{-j(N/2)\omega}\cos(N\omega/2) \tag{5}$$

This second case is likely familiar, and one of the type which we are likely to do instinctively.

Note that our goal here is to bring out a linear phase term [or a linear phase plus 90° as in equation (4)]. As such, the solution to the problem, that of obtaining α and β, can be viewed as choosing α to be the delay in the middle, equation (2a), while β is the lead or lag relative to this middle, equation (2b). This is in fact the very best way to remember how to do this quickly. Equations (4b) and (5) can be quickly developed keeping the method in mind.

Another extension is the case of many equal magnitude terms. For example, we might want to sum:

$$H(e^{j\omega}) = 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega} + e^{-6j\omega} + e^{-7j\omega} \tag{6}$$

This could be summed in pairs starting with the outside terms. Indeed, we will do this below in equation (10). But, once again, there are some simplifications. Thus it is a standard trick to multiply both sides of equation (6) by some factor such as $1 - e^{-j\omega}$.

$$\begin{aligned}(1 - e^{-j\omega})\,H(e^{j\omega}) = {} & 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega} + e^{-6j\omega} + e^{-7j\omega} \\ & - e^{-j\omega} - e^{-2j\omega} - e^{-3j\omega} - e^{-4j\omega} - e^{-5j\omega} - e^{-6j\omega} - e^{-7j\omega} - e^{-8j\omega}\end{aligned} \tag{7}$$

and we clearly see that all the middle terms on the right side of equation (7) cancel, leaving:

$$H(e^{j\omega}) = (1 - e^{-8j\omega}) \;/\; (1 - e^{-j\omega}) \tag{8}$$

Here both the numerator and the denominator of equation (8) yield to the attack suggested above such that

$$H(e^{j\omega}) = e^{-4j\omega}(e^{4j\omega} - e^{-4j\omega}) \;/\; e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2}) \tag{9a}$$

$$= e^{-3.5j\omega}\,[\,\sin(4\omega) \;/\; \sin(\omega/2)\,] \tag{9b}$$

Here, two factors of 2j from the Euler relationships have cancelled. Note that the total phase delay is still, the middle delay of all the terms summed. The result is the so-called Dirichlet function, sometimes also called a periodic sinc or an aliased sinc.

By a useful regrouping of the terms of equation (6) we can obtain pairs that can then be reduced to cosines, with all terms having the same linear phase:

$$H(e^{j\omega}) = (1 + e^{-7j\omega}) + (e^{-j\omega} + e^{-6j\omega}) + (e^{-2j\omega} + e^{-5j\omega}) + (e^{-3j\omega} + e^{-4j\omega}) \tag{10a}$$

$$H(e^{j\omega}) = e^{-3.5j\omega}(e^{3.5j\omega} + e^{-3.5j\omega}) + e^{-3.5j\omega}(e^{2.5j\omega} + e^{-2.5j\omega})$$
$$+ e^{-3.5j\omega}(e^{1.5j\omega} + e^{-1.5j\omega}) + e^{-3.5j\omega}(e^{0.5j\omega} + e^{-0.5j\omega}) \qquad (10b)$$

$$H(e^{j\omega}) = 2\, e^{-3.5j\omega} [\, \cos(0.5\omega) + \cos(1.5\omega) + \cos(2.5\omega) + \cos(3.5\omega)\,] \qquad (10c)$$

This is (as can be shown using trig identities) identical to the result of equation (9b). Note that this even length filter has a frequency response that is the sum of cosines that have arguments that are odd integer multiples of $\omega/2$.

At this point, it is a simple matter to address more general frequency response functions. What we have been doing above is actually finding closed form expressions for DTFT summations of the form:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-jn\omega} \qquad (11)$$

where h(n) is the impulse response and $H(e^{j\omega})$ is the frequency response. Accordingly equation (6) and the summation results of either equation (9b) or equations (10c) represent a filter with impulse response h(n)=1 for n=0 to n=7, and h(n)=0 otherwise. This is the so-called "rectangular window."

Suppose that h(n) is more, but not completely general: of finite length N, and symmetric. For example, h(n) = h(N-1-n) for n=0 to n=N-1, and h(n)=0 otherwise. We can easily modify equation (6) for a length N=8 example of this situation.

$$H(e^{j\omega}) = h_0 + h_1e^{-j\omega} + h_2e^{-2j\omega} + h_3e^{-3j\omega} + h_3e^{-4j\omega} + h_2e^{-5j\omega} + h_1e^{-6j\omega} + h_0e^{-7j\omega} \qquad (12)$$

With this new impulse response, the procedure of equation (10) is easily modified as:

$$H(e^{j\omega}) = 2\, e^{-3.5j\omega} [\, h_3\cos(0.5\omega) + h_2\cos(1.5\omega) + h_1\cos(2.5\omega) + h_0\cos(3.5\omega)\,] \qquad (13)$$

This makes the point, by example, that the DTFT is a Fourier Series with the usual roles of time and frequency reversed. That is, a frequency response of a discrete-time system (a time description that is discrete) is continuous and periodic.

A similar (slightly simpler?) result is obtained for an odd length impulse response. For example:

$$H(e^{j\omega}) = h_0 + h_1e^{-j\omega} + h_2e^{-2j\omega} + h_1e^{-3j\omega} + h_0e^{-4j\omega}$$

$$= e^{-2j\omega} [\, h_2 + 2h_1\cos(\omega) + 2h_0\cos(2\omega)\,] \qquad (14)$$

which is a cosine series (again, a Fourier Series dual) with arguments that are all integer multiples of ω.

The frequency response for odd symmetry impulse response is also similar. For odd symmetry an example would be:

$$H(e^{j\omega}) = h_0 + h_1e^{-j\omega} + h_2e^{-2j\omega} - h_1e^{-3j\omega} - h_0e^{-4j\omega}$$

$$= e^{-2j\omega}\{ h_2 + 2e^{j\pi/2}[h_1\sin(\omega) + h_2\sin(2\omega)] \} \qquad (15)$$

We note the phase shift of $\pi/2$ that results from the j in the Euler relationship when sin is involved [compare with equation (4b)]. This case of odd length and even symmetry is seldom useful or even encountered. For even length, odd symmetry, we get a very useful result, and an example would be:

$$H(e^{j\omega}) = h_0 + h_1e^{-j\omega} + h_2e^{-2j\omega} - h_2e^{-3j\omega} - h_1e^{-4j\omega} + h_0e^{-5j\omega}$$

$$= 2\, e^{-2.5j\omega}e^{j\pi/2}[h_2\sin(0.5\omega) + h_1\sin(1.5\omega) + h_0\sin(2.5\omega)] \qquad (16)$$

The result of equation (16) is a familiar and useful case with a 90° phase shift, and can lead to workable Hilbert transformers.

What about a general impulse response h(n)? Well, any impulse response can be easily written as the sum of an even symmetry impulse response and an odd symmetry impulse response. For example, a length 5 impulse response with values [1 2 1 5 3] has neither even nor odd symmetry. To get the even component, we reverse the sequence, add this reversal to the original, and divide by 2. To get the odd component, we reverse the sequence, subtract it from the original, and divide by 2. We would thus have:

$$[\,1\;\;2\;\;1\;\;5\;\;3\,] = [\,2\;\;7/2\;\;1\;\;7/2\;\;2\,] + [\,-1\;\;-3/2\;\;0\;\;3/2\;\;1\,] \qquad (17)$$

Since the DTFT is of course linear, we can write the frequency response as the sum of two terms - a combination of the results for even and odd symmetry as obtained above.