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LOW-PASS TO HIGH-PASS TRANSFORMATIONS: ANALOG AND DIGITAL APPROACHES

0. INTRODUCTION

The practice of teaching filter design within a digital signal processing course often involves low-pass filter examples to the virtual exclusion of all other types. Indeed, the idea that low-pass prototypes may be converted to other types (high-pass, band-pass, band-reject, etc.) is perhaps so well known that we neglect to mention it! There are numerous ways in which low-pass filters can be converted to high-pass filters, some intuitive, and some formal.

1. ANALOG LOW-PASS TO HIGH-PASS

When we speak of an analog filter, it is probably either a passive filter (with resistors R, capacitors C, and inductors L) , or an R-C active filter that we are talking about. It is often the case with an R-C active filter (a network with only resistors, capacitors, and op-amps) that one need merely switch the positions of appropriate resistors and capacitors to convert low-pass to high-pass.

A couple of simple examples will help make this general procedure clear. We can start with the simple R-C low-pass as seen in Fig. 1. (The op-amp follower is added to correspond as closely as possible to actual practice.) In the simplest of terms, this is a low-pass because the capacitor shorts high frequencies to ground. Analytically, we can derive the transfer function of the network:

$$V_{out}(s)/V_{in}(s) = T_L(s) = (1/sC) / (R + 1/sC) = 1/(1+sRC) \qquad (1)$$

which is clearly a first-order low-pass.

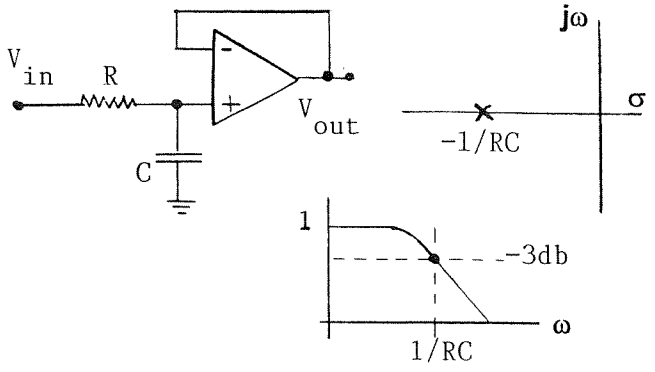
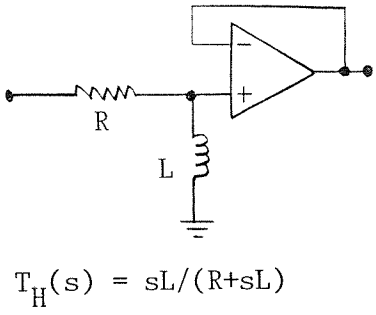


Fig. 1 A first-order low-pass having a pole at -1/RC in the s-plane and -3db frequency at 1/RC

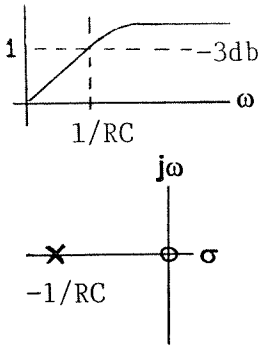
One method of obtaining a high-pass would be to substitute an inductor for the capacitor of Fig. 1 (giving Fig. 2a). Here we would say that the inductor shorts low-frequencies to ground. The analysis is similar to that of Fig. 1, keeping in mind that the impedance of the inductor is sL. Since we are usually interested in keeping inductors out of our circuits, it is useful to divide both the impedances in the voltage divider of Fig. 2a by s, effective making the resistor a capacitor and the inductor a resistor (Fig. 2b). This network has a transfer function:

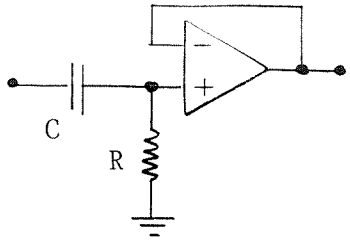
$$T_H(s) = sRC/(1+sRC) \qquad (2)$$

which has the same pole as the low-pass (at -1/RC) but now a zero at s=0 as well.



$T_H(s) = sL/(R+sL)$





$T_H(s) = sCR/(1+sCR)$

Fig. 2a

Fig. 2b

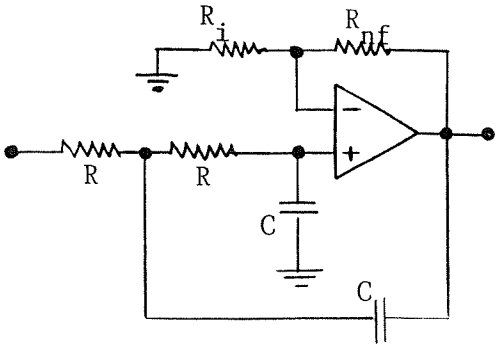
The comparison of Fig. 1 and Fig. 2b is an instance of how resistors and capacitors can be interchanged to change low-pass to high-pass. A second example is offered by Fig. 3a and Fig. 3b which are second-order active networks for low-pass and high-pass respectively. This structure is often referred to as a "Sallen-Key" filter or a "Positive Gain Voltage-Controlled Voltage-Source - Positive gain VCVS." Note that the VCVS is a fancy name for a voltage amplifier of gain K = $1+R_{nf}/R_i$. Here while resistors and capacitors are interchanged, note that the resistors in the voltage divider of the VCVS of course remain resistors. The two transfer functions are:

$$T_L(s) = (K/R^2C^2) / [\, s^2 + (3-K)s/RC + 1/R^2C^2 \,] \qquad (3a)$$

$$T_H(s) = Ks^2 / [\, s^2 + (3-K)s/RC + 1/R^2C^2 \,] \qquad (3b)$$

In our change from Fig. 1 to Fig. 2a, we substituted, in effect, s for 1/s, an inductor for a capacitor. It is important to note that while we generally interchange resistors and capacitors in practice, we are still substituting s for 1/s, since both elements (resistors and capacitors) are changed by opposite powers of s. Accordingly, we can think of a mapping of low-pass to high-pass poles as:

$$s_H = \omega_0^2/s_L \qquad (4)$$



$K=1+R_{nf}/R_i$

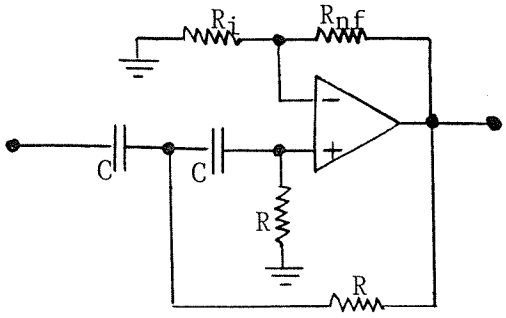


Fig. 3a Sallen-Key Low-Pass Fig. 3b Sallen-Key High-Pass

For the moment, note that the ω_0^2 is necessary just to make the units come out right. We shall see shortly that this is important for identifying the exact response relationships between the low-pass and corresponding high-pass. So far, we have just said that substituting elements gives us a high-pass rather than a low-pass. In fact, for first- and second-order filters, as we have used above so far, the type-conversion and associated responses are uncomplicated. The pairing remains uncomplicated for higher order filters only if the filter's response is Butterworth (because all the poles of the Butterworth are on the same radius). All this should become apparent as we proceed below.

Given a set of analog low-pass poles, we want to find the corresponding analog high-pass poles using equation (4). An analog low-pass pole would be at $s_L = \sigma_L + j\omega_L$ so the corresponding high-pass pole would be at:

$$s_H = \omega_0^2/(\sigma_L+j\omega_L) = [\omega_0^2/(\sigma_L^2+\omega_L^2)](\sigma_L - j\omega_L) \qquad (5)$$

Note the following: First, the parameter ω_0 is going to be the same for all poles corresponding to the particular filter. Second, the radii of the low-pass poles, $(\sigma_L^2+\omega_L^2)^{1/2}$ for each pole, can in general be expected to be different. (Butterworth filters are common and popular, and are the exception here. For Butterworth, all $(\sigma_L^2+\omega_L^2)^{1/2}$ will be the same, and usually we take $\omega_0=(\sigma_L^2+\omega_L^2)^{1/2}$ since this is the -3db or "half power" frequency for Butterworth).

To get a better idea of how the transformation of equation (5) should work, consider the following example. We start with an order 5 analog Butterworth low-pass with poles on a unit radius, and then reduce the real part of its poles by a factor of 0.22, giving Chebyshev poles for a filter with a ripple approaching 0.8 (see Fig. 4):

$$\begin{aligned} &-0.0680 + 0.9511j \\ &-0.1780 + 0.5878j \\ &-0.2200 \\ &-0.1780 - 0.5878j \\ &-0.0680 - 0.9511j \end{aligned} \qquad (6)$$

We then choose $\omega_0^2=2$ so that the high-pass poles are, from equation (5):

$$\begin{array}{l} -0.1496 - 2.0922j \\ -0.9438 - 3.1168j \\ -9.0909 \\ -0.9438 + 3.1168j \\ -0.1496 + 2.0922j \end{array} \qquad (7)$$

The corresponding low-pass and high-pass transfer functions are, respectively:

$$T_L(s) = 1 / (s^5 + 0.7119s^4 + 1.4429\, s^3 + 0.6685s^2 + 0.4254s + 0.0754) \qquad (8a)$$

$$T_H(s) = s^5 / (s^5 + 11.28s^4 + 35.45s^3 + 153.02s^2 + 151.00s + 424.19) \qquad (8b)$$

The high-pass response is seen in Fig. 4.

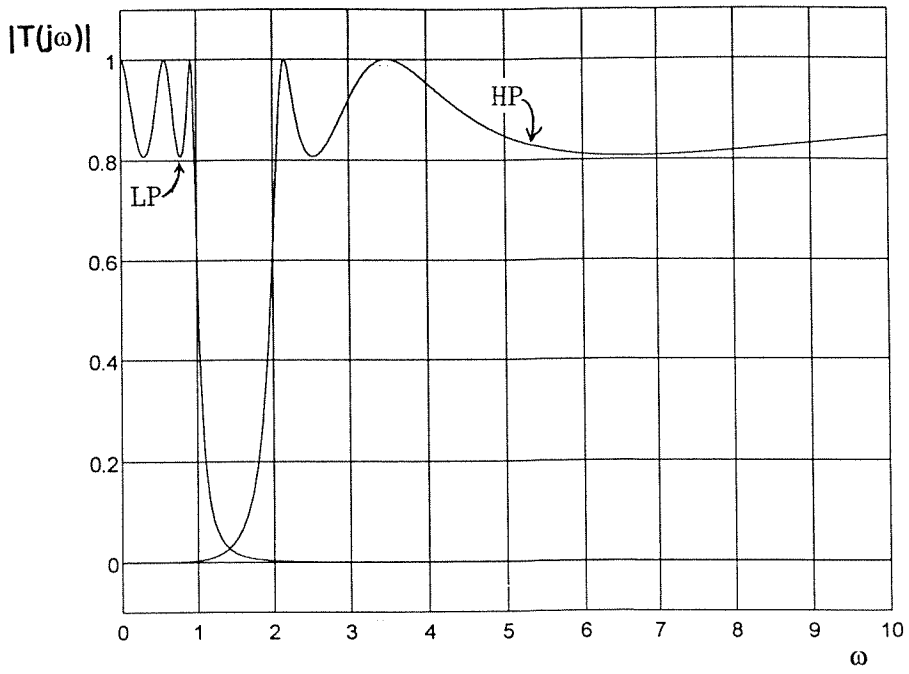


Fig. 4 A 5th-order Low-Pass is converted to a 5th-order High-Pass. Here $\omega_0 = \sqrt{2}$ and this is indeed the frequency where the two responses cross. In a practical case, we might better choose $\omega_0 = 1$, since the cutoff frequency (-3db) is close to 1. Here to avoid clutter we have chosen a higher value.

2. DIGITAL LOW-PASS TO HIGH-PASS: INTUITIVE METHODS

One intuitive method to try, for converting a known low-pass design to a corresponding high-pass, would be to subtract the low-pass from 1. That is:

$$H_H(z) = 1 - H_L(z) \tag{9a}$$

so that:

$$h_H(n) = Z^{-1}\{1\} - h_L(n) = \delta(n) - h_L(n) \tag{9b}$$

This provides a simple recipe for conversion. An alternative method to try would be to modulate the response so that it is centered about half the sampling frequency rather than about zero. This we can do by multiplying $h_L(n)$ by $(-1)^n$. An alternative way of viewing this is that the z-plane is flipped, or turned over, rotated about the imaginary axis. The connection is:

$$h_H(n) = (-1)^n h_L(n) \tag{10a}$$

so, looking at the z-Transform we have:

$$H_H(z) = \sum_{n=-\infty}^{\infty} h_H(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} (-1)^n h_L(n)\, z^{-n} =$$

$$\sum_{n=-\infty}^{\infty} h_L(n)\, (-z^{-1})^n = H_L(-z) \tag{10b}$$

from which, we easily see the flipping of the z-plane, substituting -z for z.

Both of these methods give us a high-pass from a low-pass, but in general, they give us different high-pass results. Fig. 5 shows the subtraction method, while Fig. 6 shows the substitution of -z for z method (multiplying by -1^n). Here we have started with a prototype length 41 Parks-McClellan low-pass $h_L(n)$ and $H_L(\omega)$, Fig. 5a and Fig. 5c. The filter was designed for a cutoff frequency of about 0.15 times the sampling frequency by declaring a "don't care" band from 0.14 to 0.16. Here we have subtracted 1 from the center tap of the low-pass, thus obtaining $-h_H(n)$ from equation (9b). This perhaps better shows the fact that the impulse response is identical to that of the low-pass except at the center value. (The magnitude response is the same for either case.) Note that the high-pass (Fig. 5d) has this same cutoff frequency (0.15). From Fig. 5b, we see that it is only the center term of the impulse that is altered (it becomes negative, in fact). Fig. 5e is the zero plot for the low-pass (one zero at +3.328 is not plotted) while Fig. 5f shows the corresponding zero plot for the high-pass (one zero at -3.328 not shown).

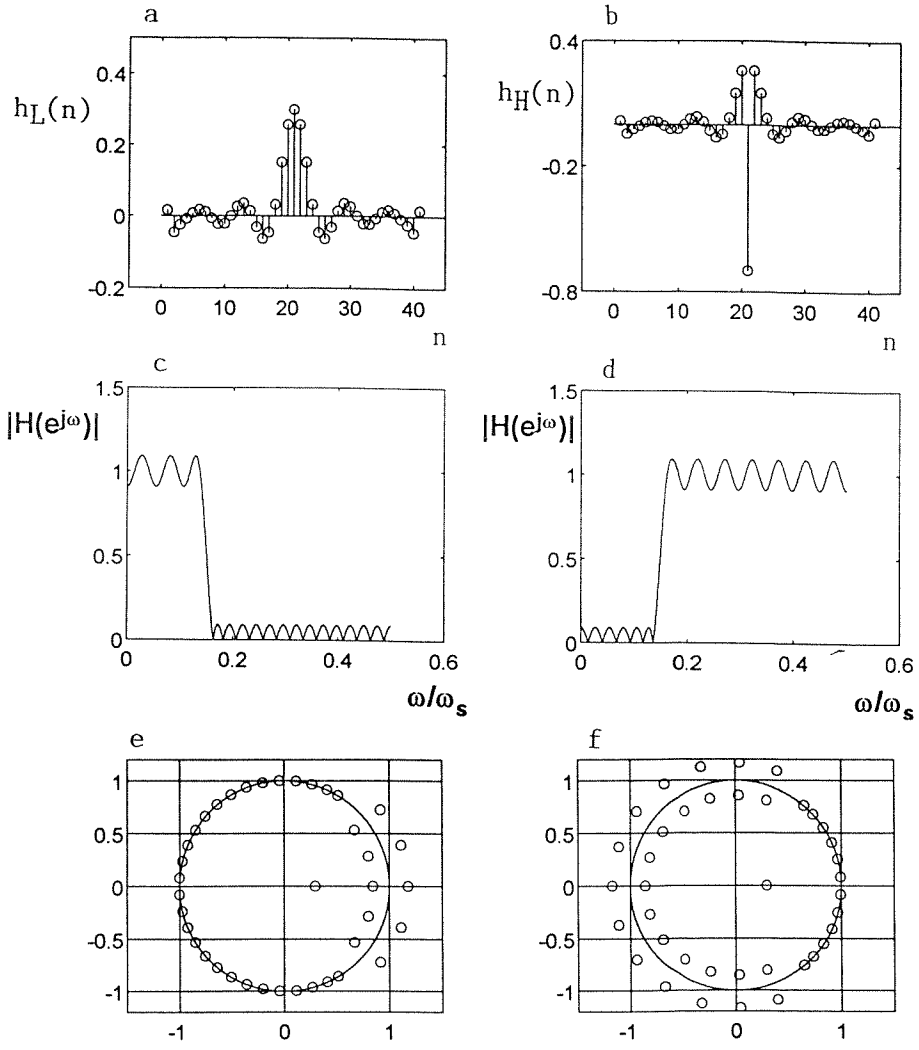


Fig. 5 In (a) we have the original low-pass impulse response, while (b) shows the high-pass impulse response (altered middle term). Here the low-pass (c) and the high-pass (d) sum to unity. The zeros (e and f) are not simply a flipping of the z-plane.

Fig. 6 in contrast has a high-pass cutoff at 0.35 the sampling frequency (see Fig. 6d) instead of at 0.15. Fig. 6a and Fig. 6c repeat the corresponding low-pass we used in Fig. 5. In Fig. 6b we see the multiplication of $h_L(n)$ by -1^n. Note that the two responses are symmetric about 1/4 of the sampling frequency. Perhaps most revealing, we see that the zero plots (Fig. 6e and Fig. 6f) are reflections of each other, across the imaginary axis.

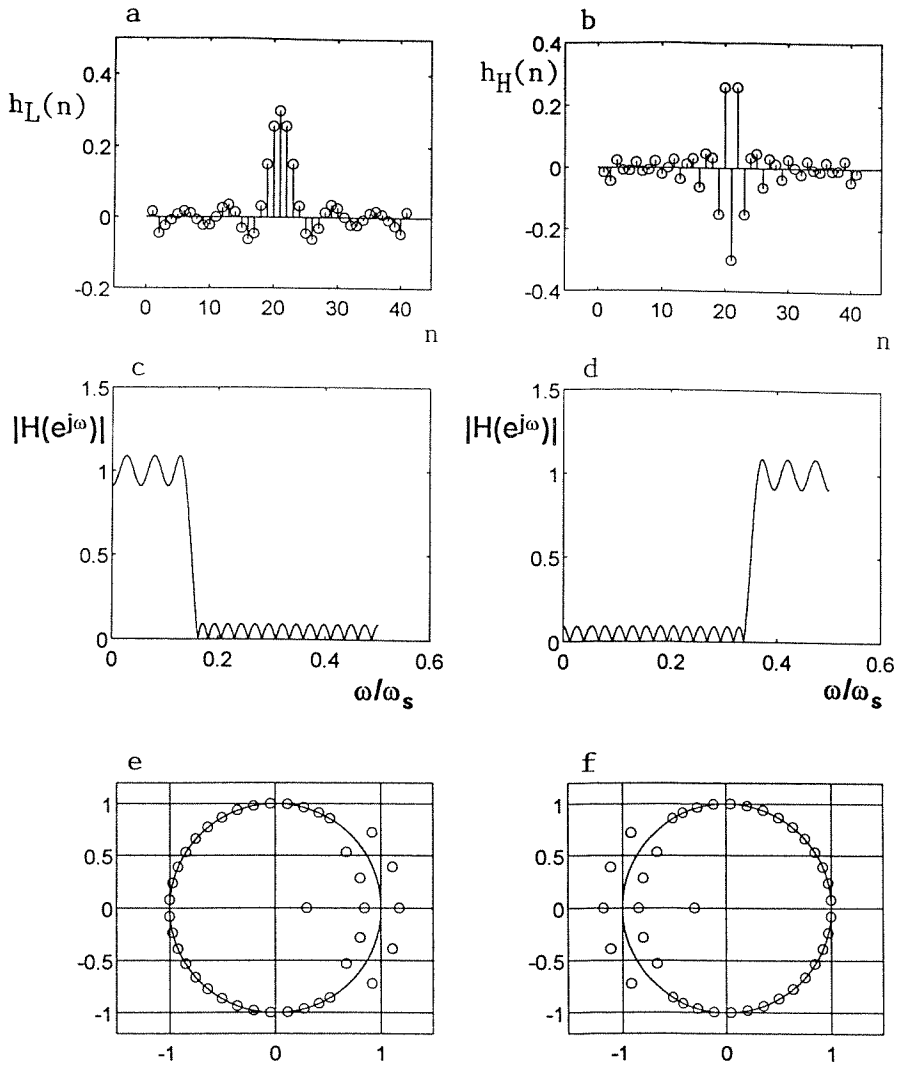


Fig. 6 In (a) we have the original low-pass impulse response, while (b) shows the high-pass impulse response (terms multiplied by -1^n). Here the low-pass (c) and the high-pass (d) are symmetric about 1/4 the sampling frequency. The zeros here (e and f) are simply a flipping of the z-plane.

Above we have applied our low-pass to high-pass tricks to FIR digital filters, and this has revealed the essential workings of these methods. While this is clearly of some accademic interest, it is perhaps less clear that we need these methods in practice. Indeed, the same programs that allow us to design low-pass FIR filters readily accept high-pass specifications (even length possibly being a problem), and many other filter specifications.

It is interesting to consider the application of the modulation technique for the case of even length. We know that we can easily have an even length low-pass, but we can't have an even length high-pass. So what happens when we take an even length low-pass prototype and convert it? If we try this, we get a perfectly good-looking magnitude response. However, the filter does not have the phase properties of the original design. For example, suppose the even order low-pass is length 4 as:

$$h_L(0) = 1/4 \qquad h_L(1) = 1/4 \qquad h_L(2) = 1/4 \qquad h_L(3) = 1/4 \tag{11}$$

which is a linear-phase FIR filter with a delay of 1.5. To get the corresponding high-pass, we get:

$$h_H(0) = 1/4 \qquad h_H(1) = -1/4 \qquad h_H(2) = 1/4 \qquad h_H(3) = -1/4 \tag{12}$$

This impulse response is anti-symmetric about a delay of 1.5, and thus represents a delay of 1.5 with an added or subtracted additional phase of 90°. The response is high-pass as expected.

The method of subtracting the response from 1 does not work well for the case of converting an even-length low-pass to an even-length high-pass. This is because this 1 is supposed to be subtracted from the center tap, and this filter has no center tap.

3. IIR FILTERS

We have above discussed conversion of low-pass to high-pass in the context of FIR filters, and suggested that this conversion may be sometimes less useful because the desired high-pass filters can often be designed directly. This continues to be true of IIR filters to some degree, although the differences here are well worth discussing.

Since most of the popular IIR digital filter design methods involve first finding a prototype analog filter, and then converting it (most often with Bilinear z-Transform method) to a corresponding digital filter, it is clear that the methods of analog low-pass to high-pass in Section 1 can be useful. It will be seen that the modulation method used for FIR filters will remain useful, although the subtraction method will not. Let's look first at the modulation method.

It remains true that we can change a low-pass to a high-pass be multiplying the impulse response by -1^n. In the FIR case, we could easily see how to apply this idea since the filter coefficients and the impulse response were one and the same. It turns out that the coefficients of an IIR filter, in both the numerator and the denominator can be alternated in sign to convert low-pass to high-pass. This is

probably not obvious until we consider that multiplying the impulse response by -1^n was the same as substituting -z for z (equations 10a and 10b). An example will be most useful here.

Lets start with a fifth-order Butterworth low-pass analog filter and reduce the real parts of the poles by multiplying them by 0.3, giving us a Chebyshev response with about 9% ripple. We then use Bilinear z-Transform to change this to a digital filter with cutoff at 0.2 times the sampling frequency (Fig. 7a).

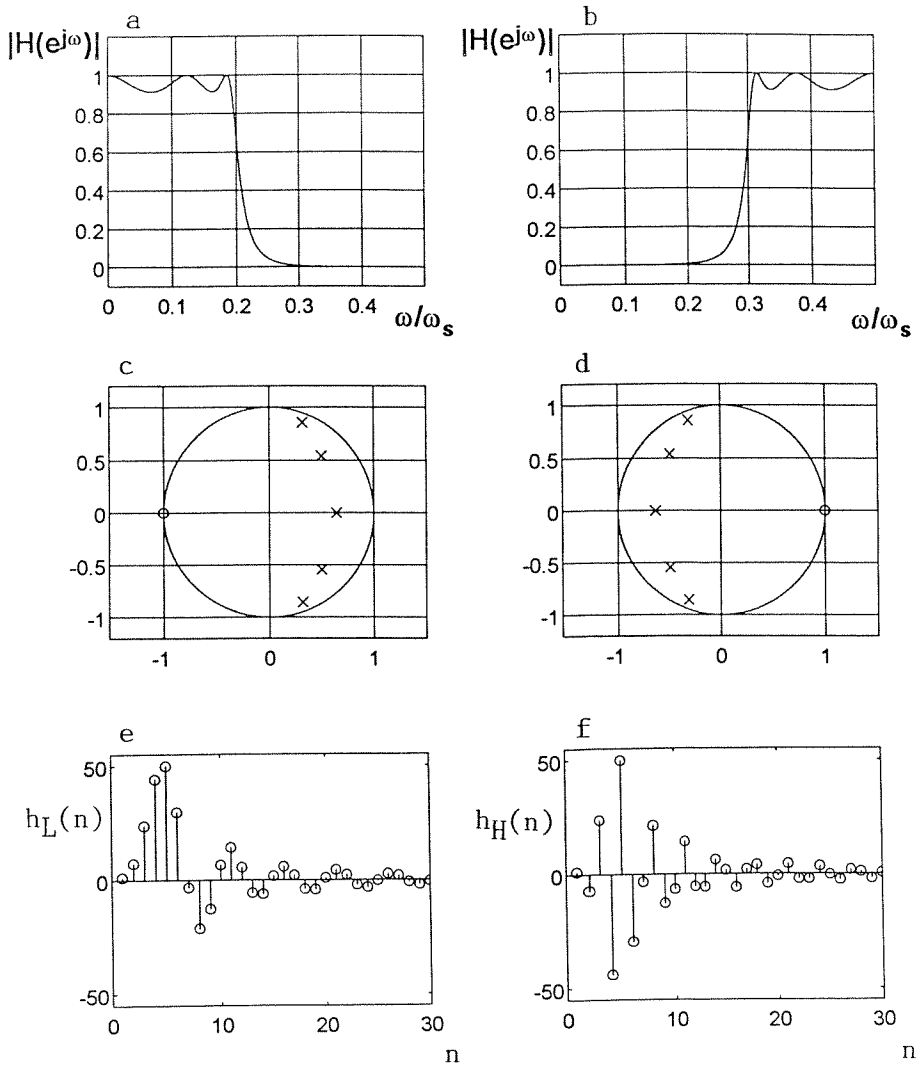


Fig. 7 In (a) we see the original low-pass magnitude response, with the pole/zero plot shown in (c) and the impulse response in (e). By changing the sign of every other coefficient in the numerator and the denominator we obtain a corresponding high-pass (b, d, and f). Indeed, every other term of the impulse response is inverted.

The transfer function is:

$$H_L(z) = \frac{(z^5 + 5z^4 + 10z^3 + 10z^2 + 5z + 1)}{z^5 - 2.2873z^4 + 3.0830z^3 - 2.4905z^2 + 1.2216z - 0.2941} \quad (13)$$

Form our discussion, the corresponding high-pass transfer function will be obtained by flipping the signs of every other term of the numerator and denominator (the starting point is abritrary - just resulting in an overall inversion).

$$H_H(z) = \frac{(z^5 - 5z^4 + 10z^3 - 10z^2 + 5z - 1)}{z^5 + 2.2873z^4 + 3.0830z^3 + 2.4905z^2 + 1.2216z + 0.2941} \quad (14)$$

This results in the magnitude response shown in Fig. 7b. By comparing the pole/zero plots (Fig. 7c and Fig. 7d) we see the filpping of -z for z. The impulse responses are also shown, computed directly from the transfer functions. We find that indeed, the high-pass has every other term inverted, relative to the low-pass.

The subtraction method, as stated above, does not work for IIR. This is because with it we would have to make special efforts to get phases to match. Put another way, the 1 from which we are subtracting $H_L(z)$ is not just a number 1, but rather a magnitude of 1 with a phase that would have to be the same as that of $H_L(z)$. This is probably unduly difficult because the phase is not linear, nor is it represented by an integer delay. In the FIR case, things did work because we generally had linear phase and a center term (at integer delay) to modify.

4. FILTER TYPE TRANSFORMATION BY SUBSTITUTING FOR Z

We are familiar with the idea of converting an analog prototype to a corresponding digital filter using Bilinear z-Transform by means of a substitution:

$$s \longleftarrow (2/T)[(z-1)/(z+1)] \quad (15)$$

A comprehensive method for converting digital filters of one type to another type by way of substituting within the z-plane is well established [1-3]. Here we will use an intuitive derivation of the form of this substitution for the digital low-pass to digital high-pass case.

To do this, let's consider the design of digital filters from analog prototypes, using Bilinear z-Transform. We will simply use the first-order low-pass (Fig. 1) and high-pass (Fig. 2b) as prototypes.

Starting with equation (1) , using equation (15) we arrive at:

$$H_L(z_L) = T_L(s) \Big|_{\,s \leftarrow (2/T)[(z_L-1)/(z_L+1)]}$$

$$= (T/2RC)(z_L+1) \;/\; [\, z_L(\, 1 + T/2RC\,) \;+\; (\, -1 + T/2RC\,)\,] \qquad (16)$$

And starting with equation (2b), we arrive at:

$$H_H(z_H) = T_H(s) \Big|_{\,s \leftarrow (2/T)[(z_H-1)/(z_H+1)]}$$

$$= \quad (z_H - 1) \;/\; [\, z_H(\, 1 + T/2RC\,) \;+\; (\, -1 + T/2RC\,)\,] \qquad (17)$$

These are familiar digital transfer functions and familiar digital filters. Note however that we have added a subscript L or H on to z. This means that we can write something that would otherwise make no sense:

$$H_L(z_L) = H_H(z_H) \qquad (18)$$

This is only possible of course if z_L and z_H are different. Equation (18) will let us find the relationship between z_L and z_H, at least for this case. It will be convenient to use the notation:

$$\alpha \equiv T/2RC \qquad (19)$$

from which equation (18) can yield:

$$z_H = [\, -z_L(1+\alpha^2) + (1-\alpha^2)] \;/\; [\, z_L(\alpha^2-1) + (1+\alpha^2)\,] \qquad (20)$$

or more simply:

$$z_H^{-1} = -\,[z_L^{-1} + a] \;/\; [1 + az_L^{-1}] \qquad (21)$$

where :

$$a = (\alpha^2-1)/(\alpha^2+1) \qquad (22)$$

Equation (21) is in fact the exact form for the substitution suggested by the formal methods [1-3].

One immediate consequence of this intuitive derivation, which we might properly infer, is that when we are considering digital and analog filters to be related by the Bilinear z-Transform, it does not matter whether we do the low-pass to high-pass transformation at the analog level or at the digital level. That is, we

might design the analog low-pass, convert it to an analog high-pass, and then Bilinear z-transform to a digital high-pass. Alternatively, we can Bilinear z-Transform the analog low-pass, and then convert the resulting digital low-pass to a high-pass using equation (21). The results will be the same.

Note that there is a special form for equation (21) for the case of a=0, which gives us:

$$z_H = -z_L \tag{23}$$

This is a valid special case, and is the modulation method that we have already discovered and used (changing the signs of every other term in the numerator and the denominator).

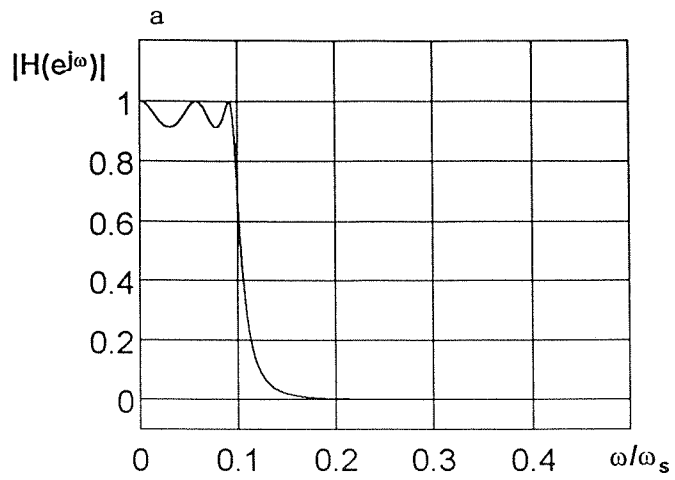
For a more general understanding of the parameter a, we will resort to the references [1-3] and some examples. Basically, the transformation in equation (21) is an all-pass transformation, and is expected to be stable as long as $|a|<1$. The references give an equation for a as follows:

$$a = -\cos[(\omega_c + \omega_c')/2] \,/\, \cos[(\omega_c - \omega_c')/2] \tag{24}$$

where ω_c is the "cutoff" frequency of the low-pass filter, and ω_c' is the "cutoff" frequency of the corresponding high-pass filter. These frequencies are defined for T=1 (f_s=1, ω_s=2π), so they are limited to the interval 0 to π. Some discussion as to what is meant by "cutoff" is probably necessary.

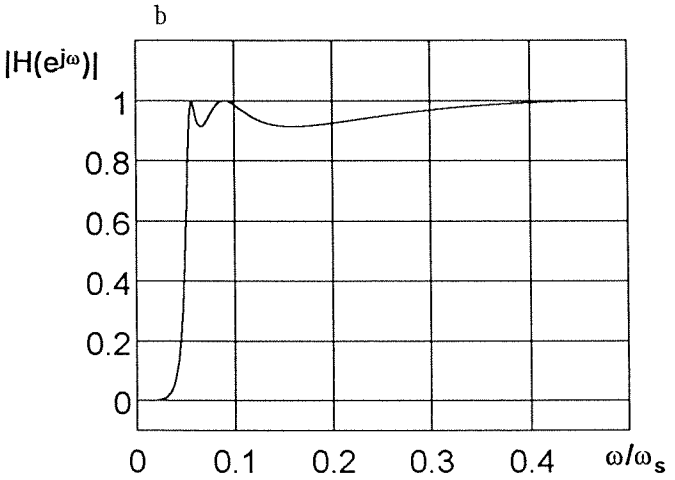
We have a good general understanding of a filter's cutoff frequency as being a frequency in the transition region between a passband and a stopband. For filters with a sharp enough cutoff region, we expect to make little error no matter how we actually define this cutoff. For example, a "half-power" point, $1/\sqrt{2}$, which is approximately but not exactly -3db is often chosen. In the cases of passband ripple, there can be some ambiguity about whether we measure down from the peaks, valleys, or some intermediate passband level. As long as the transition is very sharp, it probably matters little which reference level we use, or even if we choose a different cutoff criterion (-10db, for example). This is reassuring.

However, with regard to the present question of finding a cutoff for equation (24), things are even more forgiving. We are actually concerned with finding a particular frequency and corresponding response feature for the low-pass, and then specifying the frequency where we want this same response feature to occur for the high-pass. It does not need to be a "cutoff" frequency, but could be any feature, the peak of a second ripple for example. In fact, this is exactly the same thing we encounter when we do a Bilinear z-Transform warping. We say we are matching a "cutoff", but it is often something else, such as a peak frequency in the case of a band-pass design. The examples below should help to make things more clear.

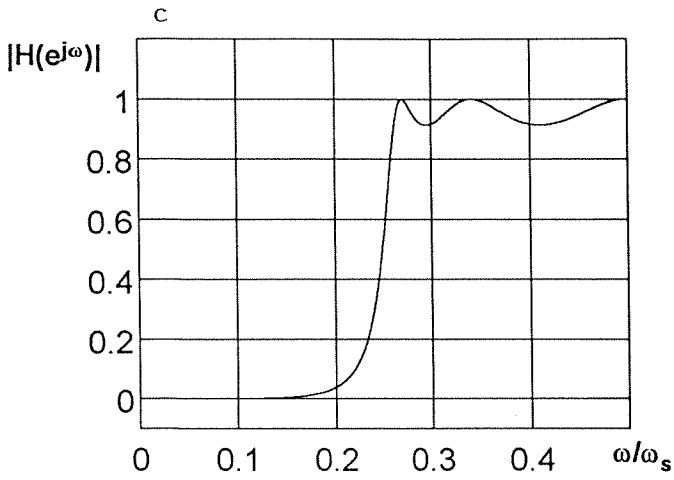


Low-Pass Prototype

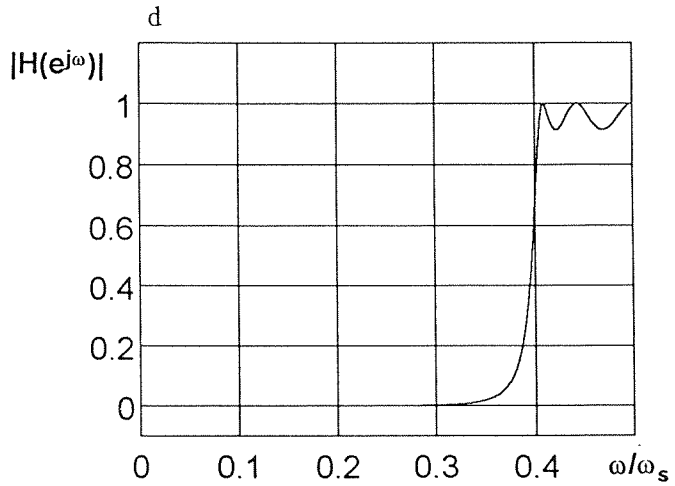
Fig. 8 The prototype low-pass (8a) is transformed to an example series of high-pass filters with increasing cutoff frequency. The a=0 case (8d) corresponds to replacing z by -z.



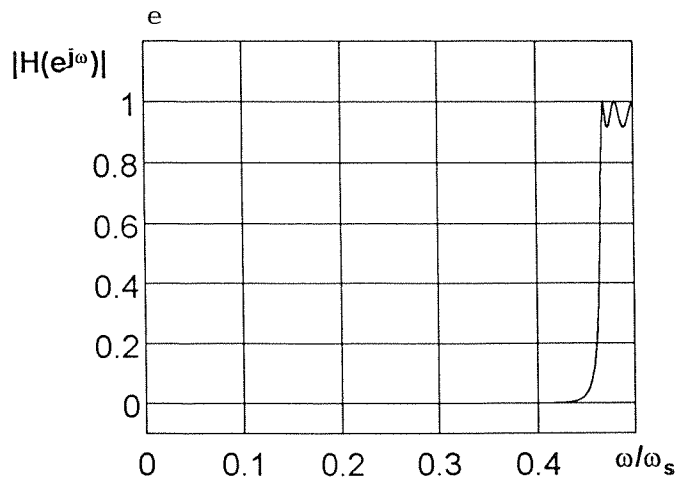
a=-0.9 High-Pass



a=-0.5 High-Pass



a=0 High-Pass



a=0.5 High-Pass

Fig. 8a shows a prototype digital low-pass (same as Fig. 7a). Figures 7b, 7d, 7e, and 7f show high-pass filters derived from the low-pass using equation (21) with values of a of -0.9, -0.5, 0, and 0.5 respectively, as indicated. This clearly shows the general progression, the cutoff frequency moving upward as a goes more positive.

To make the point with regard to "cutoff" being somewhat arbitrary in definition, note that we can choose the cutoff as magnitude 0.7 for example, and plug the numbers back into equation (24). Using Figures 8a and 8c for example, we get ω_c=0.1 $\cdot 2\pi$ and ω_c'=0.257$\cdot 2\pi$ and we find that a is back-calculated from the graph as -0.493, close to the original value of -0.5. Alternatively, suppose we choose as the "cutoff" frequency the top of the second ripple, at about 0.06$\cdot 2\pi$ in Fig. 8a and at about 0.34$\cdot 2\pi$ for Fig. 8d. Again plugging into equation (24) we get a = -0.485, again close to -0.5.

Note that as long as the high-pass response is symmetric about π exactly as the low-pass is symmetric about 0, we get $\omega_c' = \pi - \omega_c$ so that a = 0 from equation (24).

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