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FREQUENCY RESPONSE OF BANDPASS FILTER

The second-order bandpass response is represented by a transfer function of the form:

$$T(s) = \frac{As\omega_o}{s^2 + (\omega_o/Q)s + \omega_o^2} \quad (1)$$

where A is a constant, Q is the filter "Q", and ω_o is the center frequency of the bandpass. The significance of A and Q will be derived from the frequency response function, $|T(s)|$ or $|T(j\omega)|$. We note first the useful trick of finding the $|T(s)|$ at the center frequency, by just substituting $s = j\omega_o$ into T(s). This means that the first term and the last term in the denominator will cancel, and then the ω_o and the remaining s will cancel, leaving:

$$|T(s)|_{s=j\omega_o} = AQ \quad (2)$$

The significance of A is as follows. At very low frequency, s is very small and will be ignored in the denominator. Using equation (1) then,

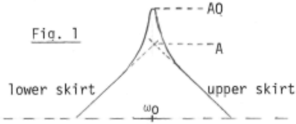
$$|T(s)| \simeq A\omega/\omega_o \qquad \text{(low-frequency)} \quad (3)$$

Similarly, at very high frequencies, relative to ω_o, only the s^2 term matters, and

$$|T(s)| \simeq A\omega_o/\omega \qquad \text{(high-frequency)} \quad (4)$$

Equations (3) and (4) thus represent the asymptotic behavior of $|T(s)|$ far away from the center frequency. Note that these represent 6db/octave slopes, the so-called "skirts" of the bandpass. These equations are certainly not valid at the center frequency, but at the center frequency, they both agree that $|T(s)| = A$. In a case where the skirts have been measured and plotted accurately on log-log paper, the lines can be extended to intersect at ω_o, and should intersect at a magnitude value of A. At the same time, the actual value at ω_o should be (equation 2) AQ. Thus we have a construction, as in Fig. 1, which could be used to obtain a measure of Q. This would be particularly useful in cases where a sharp bandpass along with skirts has been measured. In such a case, reading the 3db bandwidth (see below) to get the Q would be very inaccurate, while the method of Fig. 1 could give excellent results.

Fig. 1



Above we have found some useful results and have so far avoided the calculation of a general from of $|T(s)|$. For some additional results, this will be necessary, and we will be using the usual:

$$|T(s)| = [T(j\omega)\cdot T(-j\omega)]^{1/2} \quad (5)$$

$$= \left[\frac{Aj\omega\omega_o}{(j\omega)^2 + \frac{j\omega_o\omega}{Q} + \omega_o^2} \cdot \frac{A(-j\omega\omega_o)}{(-j\omega)^2 - \frac{j\omega_o\omega}{Q} + \omega_o^2}\right]^{1/2} \quad (6)$$

$$= \left[\frac{A^2\omega^2\omega_o^2}{(\omega_o^2 - \omega^2 + j\omega_o\omega/Q)\ (\omega_o^2 - \omega^2 - j\omega_o\omega/Q)} \right]^{1/2} \tag{7}$$

$$= \left[\frac{A^2\omega^2\omega_o^2}{(\omega_o^2 - \omega^2)^2 + \omega_o^2\omega^2/Q^2} \right]^{1/2} \tag{8}$$

$$= \left[\frac{A^2Q^2}{1 + Q^2\frac{(\omega_o^2-\omega^2)^2}{\omega_o^2\omega^2}} \right]^{1/2} \tag{9}$$

$$= \left[\frac{A^2Q^2}{1 + Q^2\left[\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega} \right]^2} \right]^{1/2} \tag{10}$$

$$= AQ\left[\frac{1}{1 + Q^2\left[\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega} \right]^2} \right]^{1/2} \tag{11}$$

Virtually any of the equations from (7) through (11) would be useful for actually plugging in numbers, but the final form in equation (11) will prove useful for our derivations. Note first that equation (11) verifies equation (2) for the response at the center frequency. Moreover, it also tells us the important fact that ω_o really is the maximum response frequency, as the second term in the denominator is zero only when $\omega = \omega_o$, and is otherwise positive, decreasing the response.

Continuing, we note that any two points on opposite sides of the center frequency, which have the same response value, must have the same second term in the equation (11) denominator. If the lower of these is ω_ℓ, and the upper ω_u, then it must be true that:

$$\left[\frac{\omega_\ell}{\omega_o} - \frac{\omega_o}{\omega_\ell}\right]^2 = \left[\frac{\omega_u}{\omega_o} - \frac{\omega_o}{\omega_u}\right]^2 \tag{12}$$

or:

$$\omega_o = (\omega_u\omega_\ell)^{1/2} \tag{13}$$

which means that the center frequency is the geometric mean of <u>any</u> two frequencies that have the same response value.

Equation (13) is certainly true for the -3db frequencies. [Actually, we mean the half power frequencies, where the response is down by $1/\sqrt{2}$ which is -3.0103... db]. If we use equation (13), we can show that the 3db bandwidth, B, is given by:

$$B = \omega_u - \omega_\ell = \omega_o\left[\frac{\omega_u}{\omega_o} - \frac{\omega_o}{\omega_u}\right] = \omega_o\left[\frac{\omega_o}{\omega_\ell} - \frac{\omega_\ell}{\omega_o}\right] \tag{14}$$

or

$$B^2/\omega_o^2 = \left[\frac{\omega_u}{\omega_o} - \frac{\omega_o}{\omega_u}\right]^2 = \left[\frac{\omega_o}{\omega_\ell} - \frac{\omega_\ell}{\omega_o}\right]^2 \tag{15}$$

But, in order for $|T(s)|$ to be $1/\sqrt{2}$ down from peak, it is necessary that:

$$\left[\frac{\omega_u}{\omega_o} - \frac{\omega_o}{\omega_u}\right]^2 = \left[\frac{\omega_o}{\omega_\ell} - \frac{\omega_\ell}{\omega_o}\right]^2 = 1/Q^2 \tag{16}$$

Combining equations (15) and (16), we obtain:

$$Q = \omega_o/B = \omega_o/(\omega_u - \omega_\ell) \tag{17}$$

or, the Q is the center frequency divided by the 3db bandwidth.