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SECOND-ORDER STEP RESPONSE

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Many systems receive a step function (or the leading edge of a long pulse or square wave) and respond with an overshoot of the top of the step followed by some degree of "ringing" before the response can be said to have settled. Such a response is found in a network that is second-order or higher. It is useful to be able to calculate this response in terms of the poles of the network. This allows us to predict the overshoot and ringing in terms of other things we may know about the network. Here we will consider a low-pass response.

A second-order system is characterized by a transfer function of the form:

$$T(s) = \frac{1}{s^2 + Ds + 1} \quad (1)$$

where D is the damping that varies from 0 to 2 as Q (= 1/D) varies from ∞ to 1/2. With D between 0 and 2, the poles of the system, as given by the quadratic formula:

$$p_{1,2} = -D/2 \pm j\sqrt{1 - D^2/4} \quad (2)$$

are complex conjugates. Such complex conjugate poles will result in overshoot. In factored form, the denominator of equation (1) can be written:

$$[s - \underbrace{(-D/2 - j\sqrt{1-D^2/4})}_{p_1}][s - \underbrace{(-D/2 + j\sqrt{1-D^2/4})}_{p_2}] \quad (3)$$

Recall that the response of a system with transfer function T(s) to an arbitrary input F(t) is:

$$X(s) = T(s) \cdot F(s) \quad (4)$$

where X(s) is the Laplace transform of the output, and F(s) is the Laplace transform of the input F(t). Consulting tables, we find that when F(t) is a step function, F(s) = 1/s, so X(s) becomes:

$$X(s) = \frac{1}{s(s-p_1)(s-p_2)} \quad (5)$$

where p_1 and p_2 are as given in equation (3) above. The remaining task is to find the actual time response - the inverse Laplace transform of X(s). Again, we consult tables, and find:

$$\frac{1}{(s-a)(s-b)(s-c)} \xrightarrow{LT^{-1}} \frac{(b-c)e^{at} + (c-a)e^{bt} + (a-b)e^{ct}}{(a-b)(b-c)(c-a)} \quad (6)$$

This we can use directly to obtain the inverse Laplace transform of equation (5). It is common notation to write $-D/2$ as α and $\sqrt{1-D^2/4}$ as β . Combining equations (3), (5), and (6), and using the relationships:

$$\sin \beta t = \frac{e^{j\beta t} - e^{-j\beta t}}{2j} \quad \cos \beta t = \frac{e^{j\beta t} + e^{-j\beta t}}{2}$$

we obtain the expression:

$$X(t) = 1 - e^{\alpha t} \cos \beta t + (\alpha/\beta) e^{\alpha t} \sin \beta t \quad (7)$$

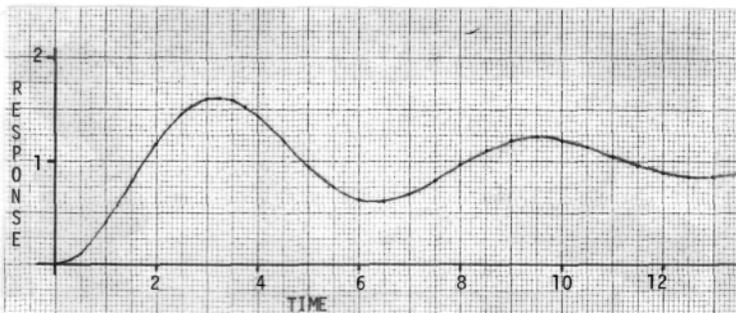
This form is one from which we can easily calculate the time waveform.

It is convenient to evaluate equation (7) using a calculator program. A suitable program using the TI-59/58 calculator is given below:

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LRN
2nd Lbl D STO 01 ÷ 2 = +/- STO 02
RCL 01 x2 ÷ 4 = +/- + 1 = √x STO 03 2nd Rad R/S
2nd Lbl A STO 04
RCL 02 × RCL 04 = INV Lnx STO 05
RCL 03 × RCL 04 = 2nd Sin × RCL 05 = STO 06
RCL 03 × RCL 04 = 2nd Cos × RCL 05 = STO 07
RCL 06 × (RCL 02 ÷ RCL 03) = STO 08
- RCL 07 + 1 = STO 09 R/S
2nd Lbl B RCL 02 × 2nd π = ÷ RCL 03 = INV Lnx × 100 = STO 10 R/S
LRN
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To use the program, one simply enters the program and then enters the value of the damping D and presses the D key. The program is now set to compute the time response. It is then a matter of choosing the time at which the response is to be calculated, entering it, and pressing the A key. Note that the calculations are normalized to one radian per second, which is one "ring cycle" in about 6.28 seconds. Thus, if you are looking for the first ring cycle, you will enter values of time from 0 to 6 for example, and larger numbers for additional settling of the ringing. An additional feature of the program is obtained by pressing the B key. This calculates the maximum value of the overshoot according to a method given by J. B. Murdock in Network Theory, McGraw-Hill (1970).* The overshoot is in percent, and corresponds to the amplitude of the first peak following the rise from zero relative to the height of the final value of the response after a very long time. Knowing the peak overshoot, it is not difficult to locate the time of the maximum by trial and error searching around 3 seconds with the A key.

As an example, we can consider $D = 0.3$ ($Q = 3.333\dots$), for which the poles are at $-0.15 \pm 0.9887j$ (at $\alpha \pm j\beta$ that is). The peak overshoot is 62.087% so we would look for a value of 1.62087 in the response. A full plot of the response is shown in the figure below. We can locate the peak around 3.18 seconds. Note that if we treat the response as a damped sinusoidal (as it is, in fact), the period of the sinusoid is about $2\pi = 6.28$ seconds. Overshoot ranges from 0 to 100% maximum.



*For other calculators, you will use the original formula given by Murdock, which in the notation here is % Overshoot = $100 \exp(\alpha\pi/\beta)$.