Experimental Electronic Music Devices Employing Walsh Functions

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The complete orthonormal set of Walsh functions is used to generate periodic waveforms and envelope shapes for an additive synthesis electronic music device. The Walsh functions, easily produced by digital circuitry, can be used to generate banks of harmonic and nonharmonic waveforms. A second Walsh function generator forms the basis of a digital envelope controller which can produce a wide variety of simultaneous envelope shapes.

INTRODUCTION: Although investigated by J. L. Walsh in 1923 [1], the set of functions which now bear his name have not found wide application until recent years [2], [3], [4]. Walsh waveforms are rectangular, taking on only the values ± 1 over a basic interval, after which the sequence may be repeated to form a set of periodic functions. Walsh functions form a complete orthonormal set and, therefore, can be employed in waveform synthesis schemes analogous to the Fourier synthesis methods which employ sines and cosines. The ± 1 levels are easily converted to the zero and one levels of digital logic, and numerous schemes for the generation of Walsh functions by digital means have been suggested [5], [6].

PROPERTIES AND GENERATION METHODS

Walsh functions indexed from zero to $2^m - 1$ are defined on a basic interval, such as zero to one, which is subdivided into 2^m equal segments, where m is an integer. The functions indexed by $2^t - 1$, where t is an integer less than or equal to m, are known as Rademacher functions, and are actually a set of square waves in octaves, starting at $+1$, and repeating a total of 2^{t-1} times in the basic interval. The remaining Walsh functions can be generated from the recursion relation:

$$Wal(h) \cdot Wal(k) = Wal(h \oplus k)$$

where the notation Wal(j) denotes the Walsh function of index j, the symbol ($\oplus$) represents modulo-2 addition ($0 \oplus 0 = 0$, $0 \oplus 1 = 1$, $1 \oplus 0 = 1$, and $1 \oplus 1 = 0$), and h and k are represented by their binary equivalents. After converting the Walsh functions to the zero and one logic levels, the indicated multiplication ($\cdot$) in the recursion relation reduces to modulo-2 addition [5] (the logical "EXCLUSIVE-OR" function), suggesting a hardware generation scheme as indicated in Fig. 1 for m = 3. $Wal(0)$ is a constant function. Extension of the generation scheme for larger m is straightforward.

COMPUTER GENERATION OF HIGHER ORDERS

Generation of Walsh functions of higher index (also referred to as higher "sequency" as defined below) is facilitated by a computer program employing the algorithm:

Step 1: Generation of the square waves in positions $2^t - 1$:

```
Repeat for k = 1, 2, 3, . . . . m
  Define L = 2^(m-k)
    Repeat for r = 1, 2, 3, . . . 2^m
      p = r/L - 1/2^m
      If (Highest integer in p)/2 = (An integer)
            Then W(2^k - 1, r) = 1
            Else  W(2^k - 1, r) = 0
    End
End
```

Step 2: Recursion Relation Implementation

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Repeat for k = 1, 2, 3, . . . m - 1
  Define S = 2^(k+1) - 2^k - 1
    Repeat for q = 1, 2, 3, . . . S
      Repeat for r = 1, 2, 3, . . . 2^m
        W(2^(k+1) - q - 1, r) = 1 if W(2^(k+1) - 1, r) ≠ W(q,r)
                              = 0 otherwise
      End
    End
End
```

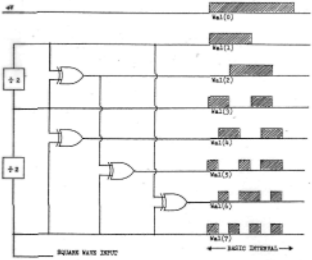


Fig. 1. Digital hardware generation of first eight Walsh functions.

A 2^m by 2^m matrix notation, where the rows of the matrix are Walsh functions in sequency order, is useful. A computer generated Walsh function matrix (W) for $m = 5$ is shown in Fig. 2.

SAL, CAL, AND SEQUENCY

Examination of the matrix in Fig. 2 indicates that successive odd and even indexed Walsh functions are shifted versions of the same sequence, and the notation $Sal(i) = Wal(2i - 1)$ and $Cal(i) = Wal(2i)$ is often used in analogy with the sine and cosine notation. The concept of frequency yields to the terminology "sequency," defined as one-half the average number of zero crossings per second (zps) [7]. The ordering of the Walsh functions in sequency order can be regarded in terms of a time-normalized sequency, where the Walsh functions are defined on an open interval so that one does not count the zero crossings at either end of the interval. Corresponding Sal and Cal functions have the same zps, and when regarded as periodic functions, they can be thought of as differing only by a time delay.

WALSH-FOURIER SERIES AND DISCRETE WALSH TRANSFORM

By analogy with the standard Fourier series employing sines and cosines, a corresponding Walsh-Fourier series can be defined [8]. Using a running variable x, the series is:

$$F(x) = \sum_{n=0}^{\infty} C_n \, Wal(n,x)$$

where

$$C_n = \int_0^1 F(x) \, Wal(n,x) \, dx.$$

Since $Wal(n,x)$ takes on only the values ± 1, it not only breaks up the integral into several subintervals of integration, but also effectively moves outside the integral sign. For a Fourier series, it would be necessary to integrate a sine or a cosine times $F(x)$ to obtain the coefficients, but for the Walsh-Fourier series, it is only necessary to be able to integrate $F(x)$. While a smooth curve will never be represented completely by a finite series of Walsh functions, a small amount of low-pass filtering is usually sufficient to remove the sharp corners of the composite waveform. Moreover, many rectangular functions such as sequences and periodically sampled analog signals can be represented exactly by a finite series of appropriately timed Walsh functions. Walsh-Fourier coefficients of such discrete sequences are obtainable as a matrix product $C = WX$, where C is a 2^m dimensioned row vector of Walsh coefficients, W is the 2^m by 2^m Walsh function matrix, as in Fig. 2 for $m = 5$, and X is a 2^m dimensioned column vector of discrete samples. This is referred to as a Discrete Walsh Transform (DWT).

FAST WALSH TRANSFORM

More economical use of computer time can be made by employing the Fast Walsh Transform (FWT) technique, derived from the Hadamard Transform and the Fast Fourier Transform (FFT) techniques [9]. The FWT follows the basic Cooley-Tukey algorithm for the FFT [10], but avoids operations with complex numbers. The FWT transforms N time-sampled data points into N discrete spectrum points. In the case where the N samples constitute a periodic waveform, or are the best approximation to a periodic waveform one could expect from N samples, the spectral points are the Walsh-Fourier coefficients that would be obtained from the DWT. The reduction in the number of computer operations is from approximately N^2 for the DWT to approximately $N \log_2 N$ for the FWT, and the matrix of Walsh functions need not be generated at all. A flow graph for the Fast Hadamard Transform (FHT) as it fits into the overall analysis and synthesis process is shown in Fig. 3 for the case of $m = 3$. The FHT yields the Walsh coefficients C_n, but in a scrambled order. The FWT is obtained by simply rearranging the coefficients in sequency order, the scrambled order being generated by the scheme shown in Table 1.

The scrambled order gives the sequencies of the rows of the Hadamard matrix. Both the flow graph and the ordering schemes are easily extended to larger order.

The FFT and FWT methods suggest possible realiza-

Wal(0)	11111111111111111111111111111111
Wal(1)	11111111111111110000000000000000
Wal(2)	00000000111111111111111100000000
Wal(3)	11111111000000001111111100000000
Wal(4)	00001111111100000000111111110000
Wal(5)	11110000000011110000111111110000
Wal(6)	00001111000011111111000011110000
Wal(7)	11110000111100001111000011110000
Wal(8)	00111100001111000011110000111100
Wal(9)	11000011110000110011110000111100
Wal(10)	00111100110000111100001100111100
Wal(11)	11000011001111001100001100111100
Wal(12)	00110011110011000011001111001100
Wal(13)	11001100001100110011001111001100
Wal(14)	00110011001100111100110011001100
Wal(15)	11001100110011001100110011001100
Wal(16)	01100110011001100110011001100110
Wal(17)	10011001100110010110011001100110
Wal(18)	01100110100110011001100101100110
Wal(19)	10011001011001101001100101100110
Wal(20)	01101001100101100110100110010110
Wal(21)	10010110011010010110100110010110
Wal(22)	01101001011010011001011010010110
Wal(23)	10010110100101101001011010010110
Wal(24)	01011010010110100101101001011010
Wal(25)	10100101101001010101101001011010
Wal(26)	01011010101001011010010101011010
Wal(27)	10100101010110101010010101011010
Wal(28)	01010101101010100101010110101010
Wal(29)	10101010010101010101010110101010
Wal(30)	01010101010101011010101010101010
Wal(31)	10101010101010101010101010101010

Fig. 2. First 32 Walsh functions generated by computer.

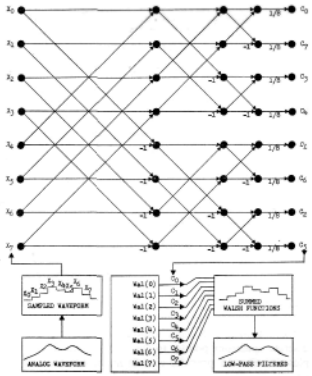


Fig. 3. Overall analysis-synthesis process showing role of Hadamard transform.

tions of polyphonic instruments, filters, and timbre controllers by feeding or controlling spectral information to the inverse transforms by means of some device such as a keyboard. The outputs of the device could be time-ordered for an audible output, and the input spectral points, roughly representing pitch information, could be enveloped "in" at the inputs.

APPLICATIONS

Periodic Waveform Generation

Using the 32 by 32 matrix of Walsh functions, as defined by Fig. 2, Walsh coefficients can be generated by DWT or FWT computer methods. Coefficients for some common waveforms are shown in Fig. **4a**. Simple operational amplifier summation techniques are used to sum the appropriate Walsh functions in proportion to their coefficients. Although sequency is not correlated with subjective musical pitch, the composite waveforms have a pitch determined by the basic interval of the Walsh functions, unless the composite waveform is specifically made to repeat more than once in the basic interval. Oscilloscope traces of the Walsh generated sine, sawtooth, and triangle approximations along with the trace of the waveforms after passing through a low-pass filter are shown in Fig. **4b**, Fig. **4c**, and Fig. **4d** respectively. In addition to common waveforms, a wide variety of complex periodic waveforms can be easily obtained. These, while interesting on an oscilloscope face, are no more interesting to the ear than the sawtooth; for example, all periodic waveforms are approximately equally boring to the ear. Low-pass filtering by fixed filters can be used to smooth the synthesized waveforms over moderate ranges of frequency. Tunable or voltage-controlled filters (VCF's) can also be employed.

WALSH HARMONIC BANK

Although the periodic waveforms are not usable directly for music synthesis (unless used with voltage-controlled amplifiers (VCA's), VCF's, etc., in a typical subtractive synthesis system), the Walsh functions can be used as a source of separated (albeit Walsh) harmonics for additive synthesis. The question of the audibility of the relative phase of the Walsh harmonic components of a composite waveform arises here; and, in general, one must allow for a different timbre depending on whether a Sal or corresponding Cal of the same sequency is employed. This is connected with the problem of monaural phase [11], and while there are cases where the phase difference is not important to tone color, it is relatively easy to devise cases where there is a great difference that can be heard when played into an open room as well as over headphones. In cases where it is possible to use less than the full set of 32 Walsh functions, the number of EXCLUSIVE-OR gates needed for the generator can be reduced (from 26 to 18) if only the Sal functions are required, and further reduced (from 18 to 11) if only one function for each sequency (zps) is to be used.

GENERATION OF ENVELOPES

Walsh functions can be used to generate envelope shapes as well as fully periodic waveforms. In this case, a suitable periodic waveform is generated over the basic interval, and can be further altered by a predetermined delay point. An envelope control circuit advances a Walsh function generator from the first segment to a predetermined stopping point, defining the attack and the sustain respectively. A restart of the generator and advance to the last segment defines the decay. Examples of two such envelopes for the $m = 5$ set of Walsh functions (32 segments) are shown in Fig. 5 for a delay at segment 16.

THE BLOCK PULSES *P*(16) and *P*(32)

In the generation of envelopes, the segment at 16

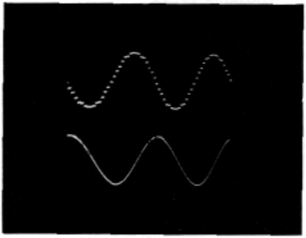
Table 1

Normal Order									Scrambled
0	Top is:	0	Y	remaining numbers: 1	Top is:	1	Z	Y	0
1	Bottom is:	7		2	Bottom is:	6			7
2	*Center:*	3		5	*Center:*	2			3
3		4		6		5			4
4								Z	1
5									6
6									2
7									5

SINEWAVE		SAWTOOTH		TRIANGLE	
a_1	0.637	a_1	-1.000	a_1	0.500
a_5	-0.264	a_3	-0.500	a_5	-0.250
a_9	-0.053	a_7	-0.250	a_{13}	-0.125
a_{13}	-0.127	a_{15}	-0.125	a_{29}	-0.063
a_{17}	-0.013	a_{31}	-0.063		
a_{21}	0.005				
a_{25}	-0.025				
a_{29}	-0.063				

NON-ZERO WALSH COEFFICIENTS IN RANGE 0 - 31

a



b

c

d

Fig. 4a. Walsh-Fourier coefficients of common waveforms. **b.** Walsh generated sinewave (top) and with low-pass filtering (bottom). **c.** Walsh generated sawtooth (top) and with low-pass filtering (bottom). **d.** Walsh generated triangle (top) and with low-pass filtering (bottom).

plays an important role. For example, if an upramp reaches its peak at 15 and falls to zero at 16, it is an attack only envelope. However, such a ramp requires the entire available set of 32 Walsh functions and hence, formidable summing problems, and would therefore be inferior to a point-by-point generation method. The ramp peaking at 16 on the other hand requires only nine of the first 32 Walsh functions, but is an attack and sustain envelope. By separating out a block pulse at segment 16, denoted $P(16)$, and using this as a blanking pulse, the attack and sustain envelope may be converted to attack only. Similarly, the easily generated downramp from segment 17 to segment 32 (decay only) can be made sustain and decay by the addition of $P(16)$. $P(16)$ is also needed as the signal-to-end attack, while the block pulse at segment 32, denoted $P(32)$, is the signal-to-end decay and go to a complete rest condition. Furthermore, $P(16)$ represents the entire sustain time, and is useful as a gate for additional effects on the signal during sustain, which otherwise would be a simple periodic waveform unacceptable for long sustain time. While $P(16)$ and $P(32)$ could be obtained as a Walsh-Fourier series, this would require summation of all of the first 32 Walsh functions. Fortunately, they are easily obtained from the Walsh functions using logic gates as indicated in Fig. 7.

OVERALL EXPERIMENTAL SYSTEM

The overall experimental system is shown in Fig. 6.

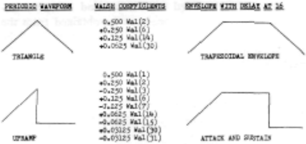


Fig. 5. Generation of envelopes from periodic waveforms by a delay at segment 16.

A voltage-controlled oscillator (VCO) is run four octaves above its normal range. Envelope control signals are obtained in conjunction with a sample-and-hold circuit, which is needed to store the frequency information during decay, i.e., after a key is lifted and the control voltage would normally disappear. Selected Walsh function harmonics are obtained from the Walsh harmonic bank, which is being driven by the VCO, and these are patched into a bank of VCA's. The VCA's are controlled by various envelopes, and the final set of amplitude shaped harmonics is mixed. The output of the mixer is then subjected to low-pass or other desired filtering by means of the VCF which can track the VCO by means of the same control voltage. The system provides time dependent harmonic changes similar to those obtained by VCF's using subtractive synthesis.

THE ENVELOPE CONTROLLER

The envelope controller is shown in Fig. 7. The attack portion of the envelope is triggered by the coinci-

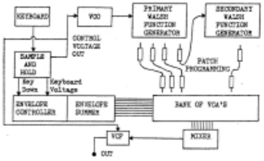


Fig. 6. Overall experimental system.

dence of a key down and a discrete change of control voltage. Thus, an attack envelope is initiated whenever the VCO changes frequency discretely, and hence a new envelope is initiated whenever a different key on the keyboard is depressed regardless of whether or not the first key is lifted completely first. Initiation of attack consists of forcing the generator from the 32 state to the first state. Upon completion of the attack, the generator is stopped by $P(16)$, and remains there until final removal of the key-down signal which forces the generator into state 17, where it is advanced by a separate decay clock to state 32. It is also fairly easy to alter the controller logic so that other modes of envelope timing can be obtained.

THE ENVELOPE SUMMER

The envelope summer is shown in Fig. 8. By adding and subtracting summed coefficients and complete envelopes, seven basic envelopes can be obtained from the envelopes of Fig. 5 plus $P(16)$. The envelopes may be low-passed to remove the sharp steps. Proper use of these envelopes gives some control over both envelope and timbre at the keyboard without touching separate controls. For example, using the trapezoidal envelope, a long attack and sustain can be obtained by always having at least one key pressed down; sharp tapping of the keys gives rapid transition to the decay state for a piano-like decay. By putting one set of Walsh harmonics under attack and sustain envelopes, and a second set under decay envelopes, the same two playing techniques will result in different voices as well as different envelopes.

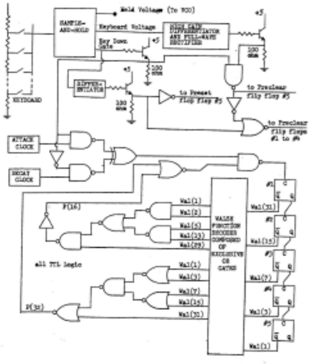


Fig. 7. Envelope controller circuit showing sources of controlling signals and method of obtaining $P(16)$ and $P(32)$ from the Walsh functions. The Walsh functions needed for the envelope summer are removed from the decoder.

NONHARMONIC TONES

The Walsh harmonic bank as discussed above produces only harmonic (in the Fourier sense) overtones, due to the fact that all the Walsh functions of the bank have the same basic interval. The experimental system has available several additional provisions for controlling

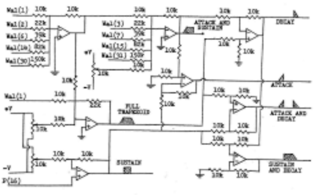


Fig. 8. Practical envelope summer. Seven basic envelopes are available. More complex envelopes require more of the Walsh functions.

timbre by introducing nonharmonic tones. Modulation effects (AM, FM and balanced) can be employed in the conventional manner to produce sidebands, thus altering timbre. Also, a second Walsh harmonic bank can be driven through a symmetric divide-by-n frequency divider (where n is not a power of two), or by a separate but tracking VCO. In the former case, the divide-by-n may be driven by any of the square-wave Walsh functions from the primary bank, and the resulting nonharmonics analyzed in terms of the frequency ratios of the square waves involved. Alternatively, the secondary bank can be driven from any of the more irregular Walsh functions. In this case, the outputs of the secondary bank will not be Walsh functions but may still be useful musically. This latter case is best analyzed in terms of a sequency triggering rate rather than through consideration of frequency ratios.

SEQUENCY TRIGGERING RATE

Suppose for example that $Wal(31)$ is actually a square wave of frequency 1600 Hz (sequency 1600 zps). $Wal(30)$ will then have a sequency of 1500 zps. If these two Walsh functions trigger identical Walsh function generators as indicated in Fig. 9, and these generators trigger on voltage transitions in one direction only, then the average triggering rate is just the sequency. The m = 5 Walsh generator involves divide-by-16 circuitry, the divide-by-16 occurring at the $Wal(1)$ output. The sequency of the $Wal(1)$ output of the first generator is therefore 100 zps, while the sequency of the $Wal(1)$

output of the secondary generator is about 93.8 zps. The divide-by-16 has in the mean time reduced the irregularity occuring in the *Wal*(30) waveform to a point where it appears as an error of one part in 16 in every sixteenth half-cycle, which may not be audible [12]. This is nearly a symmetric square wave of frequency 93.8 Hz, so the interval between the *Wal*(1) outputs of the two generators is nearly a just semitone. Analysis of the higher sequency outputs of the secondary generator becomes increasingly difficult as irregularities are not reduced by high integer division. It can be observed that single unit changes in the sequency triggering rate produce large changes in tone color.

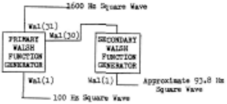


Fig. 9. Sequency triggering rate example showing the approximation of a just semitone.

The sequency drive method can be used, followed by high-integer division circuits to give relatively pure square waves [13] with frequencies proportional to the product of the driving sequency and the output sequency. Various scales and intervals can be investigated by this means. However, the primary interest in the sequency triggering rate is for the nonharmonic effects achieved.

SUMMARY

The use of Walsh functions permits an inexpensive realization of an additive synthesis system through digital-envelope control and digital harmonics. A wide variety of fixed envelopes can be obtained with minimal summation circuitry. Generation of both harmonic and nonharmonic tones using Walsh function generators serves to provide a bank of available waveforms for the additive synthesis process. Careful selection and setting of amplitudes results in sounds with a relatively strong sense of pitch, but the interval of overall periodicity may be greatly extended, and this, along with the time dependent harmonic content of the transients is more demanding on the listener's ear, and hence more demanding of the listener's attention.

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